Sharing the Surplus in Games with Externalities within and across Issues

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Abstract

We consider environments in which agents can cooperate on multiple issues and externalities are present both within and across issues. We propose a way to extend (Shapley) values that have been put forward to deal with externalities within issues to games where there are externalities within and across issues. We characterize our proposal through axioms that extend the Shapley axioms to our more general environment.

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1 Introduction

A central question in Game Theory is how to share the joint surplus among players when they cooperate. For *games in characteristic form* where the worth of a coalition depends only on the composition of this coalition, Shapley (1953) uses an axiomatic approach to characterize the unique value or payoff allocation that satisfies the properties (axioms) of efficiency, linearity, anonymity, and dummy player. This value can also be seen as an operator that assigns an expected marginal contribution to each player in the game with respect to a uniform distribution over the set of all permutations on the set of players. Alternatively, the Shapley value can be obtained as the sum of the dividends that accrue for a player from the various coalitions in which he could participate (Harsanyi, 1959) and through the potential approach proposed by Hart and Mas-Colell (1989).

Even though the Shapley value possesses many desirable properties and has inspired a host of studies, it cannot be applied to situations where externalities are present. In many economic situations, the worth of a coalition of players depends not only on the members of that coalition but also on how the rest of the players are organized. For example, in the context of international trade, the welfare of a trade union depends on whether the outside countries form other trade unions; in an oligopolistic market the profits of a cartel depend not only on the composition of this cartel but also on the organization of other firms in the market. As a natural extension of the games in characteristic form, Thrall and Lucas (1963) introduced the *games in partition function form* in which the worth of a coalition is determined by the partition of the remaining players. Using the axiomatic approach, a number of authors have proposed extensions of the Shapley value for games in partition function form. Contributions in this line of research include the works of Myerson (1977), Bolger (1989), Feldman (1996), Albizuri, Arin and Rubio (2005), Macho-Stadler, Pérez-Castrillo and Wettstein (2007), Pham Do and Norde (2007), Dutta, Ehlers and Kar (2010), and McQuillin (2009).[1]

The worth of a coalition in a partition function game depends only on the organization of all players in this game. Thus, if different games correspond to different issues under

[1] Macho-Stadler, Pérez-Castrillo and Wettstein (2006) provided mechanisms that implement a family of extensions of the Shapley value for games in partition function form.

consideration, then it is taken for granted that all issues are independent. However, there are interesting economic situations where the right approach is to consider several issues simultaneously because these issues are linked such that the amount a coalition receives in one issue depends on the way all the players are organized with respect to the other issues. In other words, in certain economic environments, there are not only multiple issues but also *externalities across issues.* For instance, consider several firms competing in multiple markets. Cooperation in one market can have an impact on the profits obtained in the other markets either through the cost functions or through the demand functions (due to product complementarities/substitutabilities). Alternatively, consider countries negotiating both a trade agreement (through, e.g., WTO) and an environmental agreement (e.g., Kyoto Protocol). These two issues, trade and environment, are linked through production. For example, the accelerated growth triggered by trade liberalization supported by the WTO is likely to raise CO2 emissions, making it more difficult for the participants in the environmental agreement to comply with their obligations under the Kyoto Protocol.

In situations like those described above, considering the issues separately, or one-by-one, is not the appropriate way to determine the values of the players. The alternative of "adding up" the two issues and then computing the value of each player also seems erroneous as it imposes that players be organized in the same way, that is, form the same coalitions, on different issues. In this paper we make a proposal that takes into account the externalities that the formation of coalitions on one issue may create on the worth of all the coalitions on the other issues.

We take the axiomatic approach and propose an extension of the Shapley value for games where there are *externalities within and across issues.* First, we present a definition of *issue-externality games*, as a natural extension of the partition form games to environments with multiple issues.[2] We consider scenarios where forming the grand coalition on all issues is the efficient outcome and worth must be allocated to the players. We then

[2] Nax (2008) proposes a similar extension that he calls multiple membership game. Nax's concern is, however, different from ours since he focuses on extending the core allocation proposed by Bloch and de Clippel (2010) for combined games (the games obtained by summing different coalitional games when bargaining over multiple independent issues) for games with externalities across issues.

propose an extension of the Shapley value to such games. Our value concept builds on a value for partition function games. That is, we extend values that have been proposed to deal with externalities within issues to environments where externalities across issues are also present.

We show that the classic axioms of linearity, player anonymity and dummy player are easily extended from the (reference) value for partition function games to issue-externality games. Also the "strong dummy property", which captures the idea that when dummy players are added to or excluded from a game the remaining players should receive the same payoffs, extends from the reference value to our proposal. In addition, we show that when the previous axioms hold for the reference value, then our extension of the Shapley value satisfies the additional properties of issue symmetry and dummy issue (which mirror for issues the ideas of dummy player and player anonymity), as well as two axioms that capture the way inter-issue externalities are considered: issue-externality anonymity and issue-externality symmetry.

Our main result is that the afore mentioned properties characterize our proposal. If a value for issue-externality games satisfies the axioms of linearity, player anonymity, strong dummy player, issue symmetry, dummy issue, issue-externality anonymity, and issue-externality symmetry, then it can be obtained as an extension (using our procedure) of a value for partition function form games that satisfies the axioms of linearity, player anonymity and strong dummy player.

This paper is organized as follows. Section 2 presents "issue-externality games" to capture externalities within and across issues. Section 3 introduces our proposed value concept. Section 4 presents the axioms. Section 5 establishes the relationship between the axioms satisfied by the value in partition function games and those satisfied by the proposed value for issue-externality games. The latter section also states our main characterization result. Section 6 illustrates our value concept through two examples and Section 7 concludes. All proofs are delineated in the Appendix.

2 The Model

In this section, we formulate "issue-externality games" with transferable payoffs that generalize partition functions games. We denote by $N = \{1, ..., n\}$ the *set of players*. A *coalition* S is a subset of players, that is, $S \subseteq N$. We denote by P a *partition* (coalition structure) of the set of players N and, for technical convenience, we follow the convention that the empty set $\emptyset$ is in P for every partition P. The *set of all partitions* of N is denoted by $\mathcal{P}$.

In our environment, players can cooperate on several issues. We denote by A the finite *set of issues* with which the players are concerned. Players can form different coalitions and partitions on different issues. Hence, to represent the way in which the players are organized, we need to specify a vector of partitions, one for each issue. Let $P^A = (P^a)_{a \in A}$ denote a vector of $|A|$ partitions of the set N, indexed by issues in A and $\mathcal{P}^A$ denote the set of vectors of $|A|$ partitions of N.[3]

An *embedded coalition* is a triplet $(S; a; P^A)$, where S is a coalition, a is an issue, and P^A is a vector of $|A|$ partitions of N such that $S \in P^a$, where P^a is the component (partition) in P^A that corresponds to issue a. An embedded coalition, hence, specifies a coalition S formed on an issue a together with the structures of coalitions formed by all the players on all issues P^A such that coalition S is an element of the structure on issue a. $ECL(N, A)$ is the set of all embedded coalitions for a given set of players N and a set of issues A.

We represent the worth that a group of players can achieve through a real-valued function $v : ECL(N, A) \to \mathbb{R}$ that associates a real number with each embedded coalition. Hence, $v(S; a; P^A)$, with $a \in A$, $S \in P^a$ and $P^A \in \mathcal{P}^A$, is the total utility available for division among members of coalition S in issue a when the players are organized on the issues in A according to the partition vector P^A. We assume that the value function satisfies $v(\emptyset; a; P^A) = 0$ for all $a \in A$ and $P^A \in \mathcal{P}^A$. The *game* (N, A, v) is called an *issue-externality game*. We denote $\mathcal{G}$ as the set of such games.

Example 1 presents a game with externalities within and across issues with two players, $N = \{1, 2\}$, and two issues, $A = \{a, b\}$.

[3] $|\Omega|$ denotes the cardinality of any set Ω.

$\downarrow a$ \ $b \rightarrow$	$\{1\},\{2\}$	$\{1,2\}$
$\{1\},\{2\}$	2, 2 3, 1	5 2, 4
$\{1,2\}$	1, 4 5	6 7

$$v(\{1,2\}; a; (\underbrace{\{1,2\},\{1\},\{2\}}_{P^A}))$$

Table 1: Example 1

We say that the game (N, A, v) has *no externalities within issues* if the worth of a coalition S on any issue a is independent of the way the rest of the players are organized on that issue. Otherwise, the game has externalities within issues. Formally,[4]

Definition 1 *The game (N, A, v) has externalities within issue $a \in A$ if for some $P^{A\backslash a} \in \mathcal{P}^{A\backslash a}$, $P^a, O^a \in \mathcal{P}$, and $S \in P^a \cap O^a$, we have $v\left(S; a; \left(P^a, P^{A\backslash a}\right)\right) \neq v\left(S; a; \left(O^a, P^{A\backslash a}\right)\right)$. The game (N, A, v) has no externalities within issues if for all $a \in A$ it is the case that $v(S; a; \left(P^a, P^{A\backslash a}\right)) = v(S; a; \left(O^a, P^{A\backslash a}\right))$ for all $P^{A\backslash a} \in \mathcal{P}^{A\backslash a}$, $P^a, O^a \in \mathcal{P}$, and $S \in P^a \cap O^a$.*

When a game has externalities within issues, the worth of a coalition on issue a depends on the organization of the other players on this issue. In a multi-issue environment, the worth of a coalition S formed on a particular issue a may depend not only on the way the rest of the players are organized on issue a but also on the organization of all players on the other issues. When this happens, we say that the game exhibits externalities across issues. More formally:

Definition 2 *The game (N, A, v) has externalities across issues if for some $a \in A$, $P^a \in \mathcal{P}$, $S \in P^a$, and $P^{A\backslash a}, O^{A\backslash a} \in \mathcal{P}^{A\backslash a}$, we have $v(S; a; (P^a, P^{A\backslash a}) \neq v(S; a; (P^a, O^{A\backslash a})$. The game (N, A, v) has no externalities across issues if for all $a \in A$, $P^a \in \mathcal{P}$, $S \in P^a$, and $P^{A\backslash a}, O^{A\backslash a} \in \mathcal{P}^{A\backslash a}$ it is the case that $v(S; a; (P^a, P^{A\backslash a}) = v(S; a; (P^a, O^{A\backslash a})$.*

[4] For notational simplicity, we use $A\backslash a$, $P^{A\backslash a}$ and $\mathcal{P}^{A\backslash a}$ instead of $A\backslash\{a\}$, $P^{A\backslash\{a\}}$ and $\mathcal{P}^{A\backslash\{a\}}$, and similarly for other sets throughout this paper.

Example 1 represents a game with externalities across issues. For instance, the stand-alone coalition of player 1 in issue a obtains a payoff of 3 if the partition in issue b is $\{\{1\},\{2\}\}$ while it obtains a payoff of 2 if the grand coalition forms in issue b.

Issues are said to be linked if there are externalities across them. Linked issues cannot be analyzed separately and must be included in the same game.

The objective of this paper is to propose a division of the surplus generated when players cooperate in an issue-externality game. We formalize the proposed division through a value. A *value* Φ specifies the payoff to players in N for any game (N, A, v), that is, a value Φ is a function from the set of games $\mathcal{G}$ to $\mathbb{R}^{|N|}$ such that $\sum_{i\in N} \Phi_i(N, A, v) = \sum_{a\in A} v(N; a; N^{|A|})$. Note that we incorporate the *efficiency axiom* into the definition of the value. We have in mind economic environments where efficiency requires that all players cooperate on all the issues, that is, $\sum_{a\in A} v(N; a; N^{|A|}) \geq \sum_{a\in A}\sum_{S\in P^a} v(S; a; P^A)$ for every vector of partitions P^A.

3 A Value for Games with Externalities within and across Issues

The class of issue-externality games $\mathcal{G}$ that we consider is quite large, encompassing partition function games as a special class. Recall that a *partition function form game* is a pair (N, u), where u is a function that associates a real number with each pair (S, P), with $S \in P$, $P \in \mathcal{P}$. That is, denoting $ECL(N) \equiv \{(S, P) \mid S \in P, P \in \mathcal{P}\}$, then $u : ECL(N) \to \mathbb{R}$. Let $PFFG$ be the set of partition function form games and denote by α a particular issue. Then $PFFG$ can be viewed as a collection of issue-externality games with a single issue, that is, $A = \{\alpha\}$, by defining $v(S; \alpha; P) \equiv u(S, P)$ for every $(S; \alpha; P) \in ECL(N, \alpha)$. Therefore, the value Φ defined for $\mathcal{G}$ also constitutes a value for $PFFG$.[5] Given that $PFFG$ encompasses the class of characteristic function games as a special case, Φ defined for $\mathcal{G}$ immediately provides a value for games in characteristic function form.

[5] We will make an assumption that ensures that $\Phi(N, \{a\}, v)$ depends only on v not on the name of the issue a chosen to represent a partition function game.

The Shapley value is one of the most important value solutions for games in characteristic form. One natural way to define a value concept for $PFFG$ is to extend the Shapley value to $PFFG$. There have been several such extensions in the literature. In the same vein, we propose value concepts for $\mathcal{G}$ by extending values defined for $PFFG$ to our broader class of games $\mathcal{G}$.

We consider a particular value ϕ^* defined for $PFFG$. We build a value concept for $\mathcal{G}$ that treats externalities across issues (i.e., inter-issue externalities) in a "similar" way as ϕ^* treats externalities within issues (i.e., intra-issue externalities). To this end, we view the contribution of each player $i \in N$ as the sum of the contributions of $|A|$ "delegates" of player i, one delegate per issue. That is, we disentangle the $|A|$ contributions of player i as if they would come from $|A|$ players. Then, we define a game with only one single issue and $|A|\,|N|$ "delegates". Finally, we apply the value ϕ to this new game.

Formally, we denote by $N(a)$ the *replica* of the set N pertaining to issue a. A typical player, coalition, and partition with respect to issue a shall be identified as $i(a)$, $S(a)$, and $P(a)$, respectively. Also, we use $N(A)$ to denote the union of all replicas of N, that is, $N(A) = \underset{a\in A}{\cup} N(a)$; hence, $N(A)$ has $|A|\,|N|$ players. For example, if $N = \{1,2,3\}$ and $A = \{a,b\}$, then $N(a) = \{1(a),2(a),3(a)\}$, $N(b) = \{1(b),2(b),3(b)\}$, and $N(A) = \{1(a),2(a),3(a),1(b),2(b),3(b)\}$. For a coalition $T \subseteq N(A)$, we denote $T(a) \equiv T \cap N(a)$. Similarly, the partition obtained by the intersection of $N(a)$ with the elements of a partition Q of $N(A)$ is denoted by $Q(a)$, that is, $Q(a) = \{T(a) \mid T \in Q\}$. $Q(a)$ is the partition of $N(a)$ as induced by Q. In our previous example, if $T = \{2(a),2(b),3(b)\}$, then $T(b) = \{2(b),3(b)\}$ and if $Q = \{\{2(a),2(b),3(b)\},\{1(a),3(a),1(b)\}\}$, then $Q(b) = \{\{2(b),3(b)\},\{1(b)\}\}$. Finally, for $T \subseteq N(A)$, $\widetilde{T}(a) \equiv \{i \in N \mid i(a) \in T(a)\}$ is the set of players whose $a-$replicas are in T, and for each $a \in A$, $\widetilde{Q}(a) \equiv \{\widetilde{T}(a) \mid T \in Q\}$ is the partition of N on issue a as induced by Q.

Given a game (N,a,v), we define the partition function form game $(N(A),\hat{v})$ as follows:

$$\hat{v}(T,Q) \equiv \sum_{a\in A} v\left(\widetilde{T}(a); a; \left(\widetilde{Q}(b)\right)_{b\in A}\right) \tag{1}$$

for any $(T,Q) \in ECL(N(A))$, that is, for any partition Q of $N(A)$ and any coalition $T \in Q$. We can think of "$\widehat{\ }$" as an operator that transforms a function from $ECL(N,A)$ to $\mathbb{R}$ to a function from $ECL(N(A))$ to $\mathbb{R}$. Such a transformation turns a game with multiple linked issues to a game with a single issue where the value of any coalition $T \subseteq N(A)$ can depend on the organization Q of all the agents.

Once (N, a, v) is transformed to $(N(A), \hat{v})$, we can apply the value ϕ^* to this game and $\phi^*_k(N(A), \hat{v})$ is the payoff for any player $k \in N(A)$. Notice that

$$\sum_{k \in N(A)} \phi^*_k(N(A), \hat{v}) = \hat{v}(N(A), N(A)) = \sum_{a \in A} v(\widetilde{N}(a); a; (\widetilde{N}(b))_{b \in A}) = \sum_{a \in A} v(N; a; N^{|A|}). \tag{2}$$

Then, we consider the sharing rule Φ^* for (N, A, v) obtained by summing, for every player $i \in N$, the payoff that all his replicas (delegates) $i(a) \in N(a)$ obtain. That is,

Definition 3 *Given a value ϕ^* for $PFFG$, we define the value Φ^* for the class of games $\mathcal{G}$ as:*

$$\Phi^*_i(N, A, v) \equiv \sum_{a \in A} \phi^*_{i(a)}(N(A), \hat{v})$$

for any game $(N, A, v) \in \mathcal{G}$.

It is immediate from (2) that the value Φ^* is efficient as long as ϕ^* is efficient. We will consider values ϕ^* for $PFFG$ that extend the original Shapley value and we will examine the properties or axioms that characterize the definition of Φ^* as given above. In the next section, we propose a list of reasonable axioms to impose on a value.

4 Axioms

We start the section with the axioms underlying the construction of the Shapley value for games in characteristic function games where there are no externalities either within or across issues. We adapt these axioms to the class of games $\mathcal{G}$. We first define the operations of addition and multiplication by a scalar, and the notions of permutation of games and dummy player.

The *addition* of two games (N, A, v) and (N, A, v') is defined as the game $(N, A, v+v')$ where $(v + v')(S; a; P^A) = v(S; a; P^A) + v'(S; a; P^A)$ for all $(S; a; P^A) \in ECL(N, A)$. Similarly, given a game (N, A, v) and a scalar $\lambda \in \mathbb{R}$, the game $(N, A, \lambda v)$ is defined by $(\lambda v)(S; a; P^A) = \lambda v(S; a; P^A)$ for all $(S; a; P^A) \in ECL(N, A)$.

Let σ_N be a permutation of N. Then the σ_N-*permutation* of the game (N, A, v) denoted by $(N, A, \sigma_N v)$ is defined by $(\sigma_N v)(S; a; P^A) = v(\sigma_N S; a; \sigma_N P^A)$ for all $(S; a; P^A) \in ECL(N, A)$, where $\sigma_N P^A$ applies the permutation σ_N to each partition P^b in P^A.

Player $j \in N$ is a *dummy player* in the game (N, A, v) if for any $(S; a; P^A) \in ECL(N, A)$ it is the case that $v(S; a; P^A) = v(S'; a; O^A)$ for any embedded coalition $(S'; a; O^A)$ that can be deduced from $(S; a; P^A)$ by changing the affiliation of player j in any issue. Hence, a dummy player j has no effect in the game: in any issue a (i) he alone receives zero for any organization of the other players; (ii) he has no effect on the worth of any coalition S; (iii) if player j is not a member of S, changing the organization of players outside S in issue a by moving player j around will not affect the worth of S, and (iv) changing the affiliation of player j in any issue other than a does not change the worth of any coalition formed on issue a.

We adapt the three original Shapley (1953) value axioms to our environment:

1. *Linearity*: A value Φ satisfies the linearity axiom if:

   1.1. $\Phi(N, A, v + v') = \Phi(N, A, v) + \Phi(N, A, v')$ for every two games (N, A, v) and (N, A, v').

   1.2. $\Phi(N, A, \lambda v) = \lambda \Phi(N, A, v)$ for any $\lambda \in \mathbb{R}$ and for any game (N, A, v).[6]

2. *Player anonymity*: A value Φ satisfies the player anonymity axiom if $\Phi(N, A, \sigma_N v) = \sigma_N \Phi(N, A, v)$ for any σ_N-permutation of (N, A, v).

3. *Dummy player*: A value Φ satisfies the dummy player axiom if $\Phi_j(N, A, v) = 0$ if player j is a dummy player in the game (N, A, v).

These three basic axioms characterize a unique value in characteristic function form games, a class of games with no externalities within or across issues (Shapley, 1953). Let (N, w) be a game in characteristic function form, where $w : 2^N \to \mathbb{R}$ is the characteristic

[6] In games without any type of externalities, additivity (part 1.1), dummy and anonymity axioms imply the property on the multiplication for a scalar (part 1.2). As shown in Macho-Stadler, Pérez-Castrillo and Wettstein (2007), in games with externalities within an issue there are values that are additive but not linear, that is, they satisfy part 1.1 (and the other basic axioms) but not part 1.2.

function. The Shapley value φ is then given by

$$\varphi_i(w) = \sum_{S \subseteq N} \beta_i(S) w(S) = \sum_{\substack{S \subseteq N \\ S \ni i}} \beta_i(S) MC_i(S) \text{ for all } i \in N, \tag{3}$$

where $MC_i(S)$ is the marginal contribution of player $i \in S$ to coalition S, that is, $MC_i(S) \equiv w(S) - w(S \backslash \{i\})$ and

$$\beta_i(S) = \begin{cases} \frac{(|S|-1)!(n-|S|)!}{n!} \text{ for all } S \subseteq N \text{ such that } i \in S \\ -\frac{|S|!(n-|S|-1)!}{n!} \text{ for all } S \subseteq N \text{ such that } i \in N \backslash S. \end{cases}$$

The three basic Shapley value axioms are compatible with many values defined for $PFFG$ and they leave an even wider leeway regarding values for games with externalities within and across issues. We now discuss some other axioms that allow us to give more structure to values in this large class of games.

First, we introduce a stronger dummy axiom that is implied by the previous three axioms in characteristic function games. Hence, it is satisfied by the Shapley value defined for this class of games but is a more demanding property than the dummy axiom when we enlarge the domain of games under consideration.

3' *Strong dummy player*: A value Φ satisfies the strong dummy player axiom if $\Phi_i(N \backslash j, A, v_{-j}) = \Phi_i(N, A, v)$ for all $i \in N \backslash j$ if j is a dummy player in game (N, A, v), where $v_{-j}(S; a; P^A) \equiv v(S \cup j; a; P^A_{+j})$ for all $(S; a; P^A) \in ECL(N \backslash j, A)$, and P^A_{+j}, with $S \cup j \in P^a_{+j}$, is similar to P^A except that player j is affiliated with some coalition in P^b for any issue b.

The strong dummy axiom states that when a dummy player is added or removed from a game, the payoffs of the remaining players do not change. This property is not satisfied by all the proposals for games with externalities within issues. The values proposed by Myerson (1977), Feldman (1996), Macho-Stadler *et al.* (2007), Pham Do and Norde (2007), de Clippel and Serrano (2008), and McQuillin (2009) satisfy the strong dummy player axiom;[7] in contrast, those of Bolger (1989) and Albizuri *et al.* (2005) do not.[8]

Next, we consider two axioms that reflect ideas akin to the player anonymity and (strong) dummy player axioms but with respect to the issues. The name of the issue should not influence the payoffs players obtain in a game, and the elimination of an issue that generates neither worth nor externalities should not change players' payoffs. We shall refer to these two axioms as issue symmetry and dummy issue, respectively.[9]

4 *Issue symmetry*: A value Φ satisfies the issue symmetry axiom if $\Phi(N, \{a\}, v) = \Phi(N, \{b\}, v')$ for any two issues a and b such that $v'(S; b; P) = v(S; a; P)$ for any $P \in \mathcal{P}$ and $S \in P$.

Thus, issue symmetry states that in a game with a single issue, renaming the issue alone does not change the value, that is, Φ depends on the game $(N, \{a\}, v)$ through v.[10]

To introduce the next axiom, we need to define the notion of dummy issue. Issue $d \in A$ is a *dummy issue* in the game (N, A, v) if $v(S; d; P^A) = 0$ for all $P^A \in \mathcal{P}^A$ and all $S \in P^a$ and $v(S; a; (P^d, P^{A\backslash d})) = v(S; a; (O^d, P^{A\backslash d}))$ for all $a \neq d$, $P^d, O^d \in \mathcal{P}$, $P^{A\backslash d} \in \mathcal{P}^{A\backslash d}$, and $S \in P^a$. Hence, no coalition can obtain any worth in a dummy issue, and the organization

[7] Among the values based on the "average approach" defined in Macho-Stadler *et al.* (2007), some satisfy the strong dummy player axiom while others do not. To illustrate this, note that all the values just mentioned but Myerson's are in the family of the average approach. To show that there are some values that do not satisfy the axiom let us define the "value alternate", which consists of applying a value in the class of average values (for example, the value proposed by Macho-Stadler *et al.*, 2007) to games with an odd number of players and another one (for example the one by de Clippel and Serrano, 2008) to games with an even number of players.

[8] These two values are not in the family of values that satisfy the average approach.

[9] For games in characteristic form, symmetry and anonymity are terms that reflect the same idea. In our environment we will keep the word anomymity to refer to properties for players and symmetry for issues.

[10] In fact, this axiom can be replaced by a stronger version. Let A and B be two sets of issues such that $|A| = |B|$ and let μ_{AB} be a bijection from the set A to the set B. Then the $\mu_{AB}-$*renaming of issues* in game (N, A, v) denoted by $(N, \mu_{AB}A, \mu_{AB}v) = (N, B, \mu_{AB}v)$ is defined by $(\mu_{AB}v)(S; b; P^B) = v(S; \mu_{AB}^{-1}(b); \mu_{AB}^{-1}P^B)$ for all $(S; b; P^B) \in ECL(N, B)$, where $\mu_{AB}^{-1}P^B$ applies the bijection μ_{AB}^{-1} to the components of the vector of partitions P^B. A value Φ satisfies the (stronger version of) issue symmetry axiom if $\Phi(N, \mu_{AB}A, \mu_{AB}v) = \Phi(N, A, v)$ for all $\mu_{AB}-$renaming of issues in (N, A, v).

of the players in a dummy issue has no effect on the worth of any coalition in any other issue.

5 *Dummy issue*: A value Φ satisfies the dummy issue axiom if $\Phi(N, A\backslash d, v_{-d}) = \Phi(N, A, v)$ if d is a dummy issue in (N, A, v) and $v_{-d}(S; a; P^{A\backslash d}) \equiv v\left(S; a; \left(P^d, P^{A\backslash d}\right)\right)$ for all $(S; a; P^{A\backslash d}) \in ECL(N, A\backslash d)$, where P^d is any partition of N.

Finally, we introduce two axioms that capture how cross-issue externalities are dealt with. The first is an axiom of anonymity on externalities across issues; it ensures that externalities across issues are treated in such a way that a player's payoff does not depend on the identity of the players exerting the externalities; rather, it depends only on the extent of the externalities.

6 *Issue-externality anonymity axiom*: A value Φ satisfies the issue-externality anonymity axiom if for any $i \in N$, it is the case that $\Phi_i(N, A, v_{\sigma_N}) = \Phi_i(N, A, v)$ for all permutations σ_N that satisfy $\sigma_N(i) = i$, where $v_{\sigma_N}(S; a; P^A) \equiv v(S; a; \left(P^a, O^{A\backslash a}\right))$ for all $(S; a; P^A) \in ECL(N, A)$, and $O^b = \sigma_N P^b$ for all $b \in A\backslash a$.

Therefore, a player's payoff does not change if the names of two other players inducing externalities from other issues are exchanged.

The second axiom pertaining to cross-issue externalities is an axiom of symmetry among issues where externalities are created. A player's payoff should not depend on the name of the issue from which externalities originate. More precisely, consider a set of players M whose only role in the game is that they induce externalities on others through their organization on one of the issues. Our issue-externality symmetry axiom then says that players' payoffs depend only on the extent of these externalities not on the issue from which players in M exert their externalities. To formulate this axiom, we first define the concept of "externality players on a single issue". Let $a \in A$. $M \subset N$ is a set of a-externality players if $v(S; b; P^A) = v(T; b; Q^A)$ for all $(S; b; P^A), (T; b; Q^A) \in ECL(N, A)$ such that

(i) $Q^c \cap (N\backslash M) = P^c \cap (N\backslash M)$ for all $c \in A$,

(ii) $Q^a \cap M = P^a \cap M$, and

(iii) $S \backslash M = T \backslash M$.

Thus, players in M affect any coalition's worth *only* through their organization on issue a, and the externalities M generates do not interact with those by $N \backslash M$. Moreover, no player in M can add to the worth of any coalition. The next axiom says that if we "transfer" the externalities exerted by M from issue a to another issue b, players' payoffs should not change.

7 *Issue-externality symmetry axiom*: A value Φ satisfies the issue-externality symmetry axiom if $\Phi_i(N, A, v_{M,ab}) = \Phi_i(N, A, v)$ for all $i \in N$ and for all set M of $a-$externality players, where the game $v_{M,ab}$ is the transformation of game v by moving the externalities induced by M from issue a to issue b, that is, $v_{M,ab}(S; c; P^A) \equiv v(T; c; O^A)$ for all $(S; c; P^A) \in ECL(N, A)$, where[11] $O^{A \backslash \{a\}} = P^{A \backslash \{a\}}$, $O^a \cap (N \backslash M) = P^a \cap (N \backslash M)$, $O^a \cap M = P^b \cap M$, and $S \backslash M = T \backslash M$.[12]

5 Characterization of the Value

In Section 3 we defined a value Φ^* for the class of games $\mathcal{G}$ by extending a reference value ϕ^* for $PFFG$. We now relate the properties of these two values. First we show that the value Φ^* satisfies a series of properties related to those satisfied by the reference value ϕ^*. Proposition 1 states that the classic axioms of linearity, player anonymity, dummy player, and strong dummy player can be extended from ϕ^* to Φ^*.

Proposition 1 *(i) If ϕ^* satisfies the linearity axiom in $PFFG$, then Φ^* satisfies the linearity axiom in $\mathcal{G}$.*
(ii) If ϕ^* satisfies the player anonymity axiom in $PFFG$, then Φ^* satisfies the player anonymity axiom in $\mathcal{G}$.
(iii) If ϕ^* satisfies the dummy player axiom in $PFFG$, then Φ^* satisfies the dummy player

[11]For $P \in \mathcal{P}$ and $S \subseteq N$, $P \cap S$ is the partition on the set S obtained from P by removing the players in $N \setminus S$.

[12]Note that for each $c \neq a$, if $S \in P^c$, then $S \in O^c$. However, it is possible that $S \in P^a$ and $S \notin O^a$. In the original game, M exerts externalities through $O^a \cap M$ while in the transformed game, M exerts externalities through $P^b \cap M$.

axiom in $\mathcal{G}$.
(iv) If ϕ^* satisfies the strong dummy player in $PFFG$, then Φ^* satisfies the strong dummy player axiom in $\mathcal{G}$.

Proposition 2 shows that when the reference value ϕ^* satisfies the strong dummy player axiom, then the properties of dummy issue and issue anonymity, which extend to issues the ideas of dummy player and player anonymity, are satisfied by the value Φ^*.

Proposition 2 *(i) Φ^* satisfies the issue symmetry axiom in $\mathcal{G}$.*
(ii) If ϕ^* satisfies the strong dummy player axiom in PFFG, then Φ^* satisfies the dummy issue axiom in $\mathcal{G}$.

Finally, Proposition 3 states that the two axioms that capture the way inter-issue externalities are considered are also satisfied given the construction of the value Φ^*, as long as the reference value ϕ^* satisfies the classic axioms of linearity and player anonymity.

Proposition 3 *(i) If ϕ^* satisfies linearity and player anonymity in PFFG, then Φ^* satisfies the issue-externality anonymity axiom in $\mathcal{G}$.*
(ii) If ϕ^* satisfies player anonymity in PFFG, then Φ^* satisfies the issue-externality symmetry axiom in $\mathcal{G}$.

Propositions 1 to 3 show that if we construct a value Φ^* for the class of issue-externality games by the procedure proposed in Definition 3, starting with a value ϕ^* for $PFFG$ that satisfies the axioms of linearity, player anonymity, and strong dummy player, then the seven axioms that we have proposed in Section 4 hold for the value Φ^*. Our main result shows that the reverse implication is also true. That is, if a value Φ for $\mathcal{G}$ satisfies the seven axioms, then it can be constructed through the proposed procedure, using a reference value ϕ for $PFFG$ that satisfies the axioms of linearity, player anonymity, and strong dummy player.

Theorem 1 *A value Φ in $\mathcal{G}$ satisfies the axioms of linearity, player anonymity, strong dummy player, issue symmetry, dummy issue, issue-externality anonymity, and issue-externality symmetry, if and only if there exists a value ϕ in PFFG that satisfies linearity,* *player anonymity, and strong dummy player such that*

$$\Phi_i(N, A, v) = \sum_{a \in A} \phi_{i(a)}(N(A), \hat{v})$$

for any game (N, A, v) *and any player* $i \in N$*, where*

$$\hat{v}(T, Q) \equiv \sum_{a \in A} v(\hat{T}(a); a; (\hat{Q}(b))_{b \in A})$$

for any partition Q *of* $N(A)$ *and any coalition* $T \in Q$*.*

In the Appendix, we show the necessity part by a sequence of steps while the sufficiency part can be deduced from Propositions 1-3.

Another property that our value concept satisfies is *independence*. To formulate this axiom, we first define the union of two issue-externality games. The union of (N, A, v) and (N, B, w) such that $A \cap B = \varnothing$ is defined as a game $(N, A \cup B, v \cup w)$ where $(v \cup w)(S; c; P^{A \cup B}) = v(S; c; P^A)$ if $c \in A$ and $(v \cup w)(S; c; P^{A \cup B}) = w(S; c; P^B)$ if $c \in B$. The axiom of independence states that players' payoffs are the same whether we analyze two games separately or the union of the two games.

8 *Independence*: $\Phi(N, A \cup B, v \cup w) = \Phi(N, A, v) + \Phi(N, B, w)$ for all games (N, A, v) and (N, B, w) such that $A \cap B = \varnothing$.

We notice that the property of independence is an axiom related to linearity, as it stipulates how the value should treat combinations of games with the same set of players. The next proposition states the exact relationship.

Proposition 4 *If a value* Φ *satisfies linearity and dummy issue, then it satisfies the independence axiom.*

It is easy to verify that the independence axiom implies the dummy issue axiom. Therefore,

Corollary 1 *Under the linearity axiom, a value* Φ *satisfies the dummy issue axiom if and only if it satisfies the independence axiom.*

6 Two Examples

Example 1. Consider a duopoly competing in two markets, a and b (see Nax, 2008 and Bulow *et al.*, 1985). Suppose that the two firms have the option to merge their operations in one or both markets. The firms' profits depend on the market structures in both markets according to the payoffs given in Figure 1 (see Section 2).

Note that we cannot analyze the two markets separately because they are linked; that is, there are externalities across the two markets. It is also inappropriate to add up the worth in the two markets. Our proposed value builds on a value for partition function games that satisfies the axioms of linearity, player anonymity, and strong dummy player. As an illustration, we use the value identified in Macho-Stadler *et al.* (2007):

$$\phi_i^{MPW}(M,u) = \sum_{(S,P)\in ECL(M)} \frac{\Pi_{T\in P\setminus S}\,(|T|-1)!}{(|M|-|S|)!}\beta_i(S)\hat{v}(S,P) \text{ for all } i \in M.$$

In our example, $M = N(A) = \{1(a), 2(a), 1(b), 2(b)\}$ and $\hat{v}$ is determined by equation (1). A straightforward computation yields

$$\phi_{1(a)}^{MPW} = 3.25, \phi_{2(a)}^{MPW} = 3.25, \phi_{1(b)}^{MPW} = 2.50, \phi_{2(b)}^{MPW} = 4.00,$$

implying that in this game, the two firms share total profits from merging in both markets as follows:

$$\Phi_1^{MPW} = 5.75, \Phi_2^{MPW} = 7.25.$$

Example 2. The class of issue-externality games $\mathcal{G}$, and the value that we propose, can accommodate situations where players meet sequentially. For example, players can meet and form coalitions at date $t = 1$ (issue a), meet again at date $t = 2$ (issue b), and the worth of the coalitions at $t = 2$ depends on the partition formed at $t = 1$. We can even consider situations where new players are active or not at $t = 2$ (that is, in issue b) depending on the coalitions formed at $t = 1$ (that is, in issue a). For example, at $t = 1$ players 1 and 2 may form a coalition or not. If they form a coalition, then player 3 participates at $t = 2$; if players 1 and 2 do not form a coalition, then players 1 and 2 are the only ones creating worth in issue b. This situation can be formalized as a game with three players and two issues where player 3 does not influence payoffs in issue a and, if players 1 and 2 do not form a coalition at $t = 1$, player 3 also does not generate any worth in issue b.

The payoffs in Figure 2 may represent such a situation. Firms 1 and 2 are initially active in the market. Firm 3 only exists if firms 1 and 2 form a coalition in issue a (that is, at $t = 1$). Therefore, the worth of any (embedded) coalition in issue a does not change if player 3 is added to or removed from it. The same happens in issue b if players 1 and 2 belong to different coalitions in issue a. On the other hand, if firms 1 and 2 are together in issue a, then firm 3 is an active player in issue b and can influence the worth obtained when he forms coalitions with either of the two players, or with both of them.

$t = 2 \rightarrow$ $\downarrow t = 1$	$\{1\},\{2\},\{3\}$	$\{1\},\{2,3\}$	$\{1,3\},\{2\}$	$\{1,2\},\{3\}$	$\{1,2,3\}$
$\{1\},\{2\},\{3\}$	$5,5,0$ $5,5,0$	$5,5$ $5,5,0$	$5,5$ $5,5,0$	$8,0$ $5,5,0$	8 $5,5,0$
$\{1\},\{2,3\}$	$5,5,0$ $5,5$	$5,5$ $5,5$	$5,5$ $5,5$	$8,0$ $5,5$	8 $5,5$
$\{1,3\},\{2\}$	$5,5,0$ $5,5$	$5,5$ $5,5$	$5,5$ $5,5$	$8,0$ $5,5$	8 $5,5$
$\{1,2\},\{3\}$	$4,4,4$ $12,0$	$2,12$ $12,0$	$12,2$ $12,0$	$10,6$ $12,0$	25 $12,0$
$\{1,2,3\}$	$4,4,4$ 12	$2,12$ 12	$12,2$ 12	$10,6$ 12	25 12

Table 2: Example 2

In Example 2, the value generated by the grand coalition is 37 and, according to the proposal ϕ^{MPW}, must be shared as

$$\begin{aligned} \phi^{MPW}_{1(a)} &= \phi^{MPW}_{2(a)} = 9.521,\ \phi^{MPW}_{3(a)} = 0 \\ \phi^{MPW}_{1(b)} &= \phi^{MPW}_{2(b)} = 6.333,\ \phi^{MPW}_{3(b)} = 5.292, \end{aligned}$$

which implies the following players' payoffs:

$$\Phi^{MPW}_1 = \Phi^{MPW}_2 = 15.854,\ \Phi^{MPW}_3 = 5,292.$$

Our proposal allows us to compute the payoff distribution from players' contributions in the different issues. The "delegates" of firms 1 and 2 in issue a obtain a total of $\phi_{1(a)}^{MPW} + \phi_{2(a)}^{MPW} = 19.042$, which is higher than the worth of 12 that they generate in that issue. Therefore, our value allocates a total worth of 7.042 to the externality that the firms' behavior in issue a generates on the value created in issue b.

Example 2 shows how to apply our values to the class of "two-stage games" proposed by Beja and Gilboa (1990). In these games, agents form a coalition in the first stage, which entitles its members to play a prespecified cooperative game at the second stage. We can think of the first stage as issue a ($t = 1$) and the second stage as issue b ($t = 2$), with the property that worth is only obtained in issue b. Beja and Gilboa (1990) characterize all the semivalues in this class of game, where semivalues satisfy linearity, player anonymity, dummy player and monotonicity. Our approach provides more structure to the values, in the sense that we propose axioms on the way externalities should be considered within and across issues (in this case, across issues, between the coalitions formed at $t = 1$ and the game played at $t = 2$). This allows, in particular, to identify the payoff that each player obtains due to his participation in each stage.

For example, Beja and Gilboa (1990) consider the following majority game. There are three players with "relative weights", or "vote counts" of $(2, 2, 3)$. If a coalition of at least two players is formed at stage 1, then the players in that coalition play a majority game to share a worth of 1. Therefore, if the coalition $\{1, 2\}$ is formed, then the two players together get 1 and each obtains a payoff of 0.5 if they do not form a coalition at $t = 2$; if the grand coalition forms at stage 1 then at stage 2 any coalition of two player obtains 1; however, player 3 ends up with a payoff of 1 in the majority game at stage 2 if either coalition $\{1, 3\}$ or $\{2, 3\}$ is formed at $t = 1$. According to the proposal ϕ^{MPW}, the worth of 1 must be shared as

$$\begin{aligned}
\phi_{1(a)}^{MPW} &= \phi_{2(a)}^{MPW} = \phi_{3(a)}^{MPW} = 0.07777 \\
\phi_{1(b)}^{MPW} &= \phi_{2(b)}^{MPW} = 0.17222,\ \phi_{3(b)}^{MPW} = 0,4222,
\end{aligned}$$

which implies players' payoffs of $\Phi_1^{MPW} = \Phi_2^{MPW} = 0.25$, $\Phi_3^{MPW} = 0,5$. The contribution of the three players to build a winning coalition in stage 1 is the same, hence they receive the same payoff 0.07777 for this contribution. However, player 3 has more power in stage

2, which is acknowledged with a payoff of $0,4222$ instead of 0.17222 for the other players.

Finally, Example 2 also suggests that the class of issue-externality games $\mathcal{G}$ can cope with situations with several linked issues where different players are "active" in each issue: the set of players is $N = N_a \cup N_b$, players in N_a take a relevant decision on issue a while N_b is the relevant set in issue b, with $N_a \cap N_b = \varnothing$. This may account for different generations of players, or different sets of countries deciding on different issues with externalities within and across them.

7 Conclusion

We have considered situations where players interact in several issues and the issues are linked because the worth of a coalition on one issue depends on the organization of the players in the other issues. We have proposed a way to extend values that have been put forward to deal with externalities within issues to games where there are externalities within and across issues. We have shown that any value for this class of games satisfies the axioms of linearity, player anonymity, strong dummy player, issue symmetry, dummy issue, issue-externality anonymity, and issue-externality symmetry, if and only if the value can be obtained as an extension of a value for partition function form games that satisfy the axioms of linearity, player anonymity and strong dummy player.

8 Appendix

Proof of Proposition 1. (i) Consider two games (N, A, v) and (N, A, v'). Since ϕ^* satisfies linearity, we have

$$\phi_k^*(N(A), \hat{v} + \widehat{v'}) = \phi_k^*(N(A), \hat{v}) + \phi_k^*(N(A), \widehat{v'}) \text{ for every } k \in N(A).$$

Also, following (1), it is easy to check that $\widehat{v + v'} = \hat{v} + \widehat{v'}$. Hence,

$$\Phi_i^*(N, A, v + v') = \sum_{a \in A} \phi_{i(a)}^*(N(A), \widehat{v + v'}) = \sum_{a \in A} \phi_{i(a)}^*(N(A), \hat{v} + \widehat{v'}) =$$
$$\sum_{a \in A} \phi_{i(a)}^*(N(A), \hat{v}) + \sum_{a \in A} \phi_{i(a)}^*(N(A), \widehat{v'}) = \Phi_i^*(N, A, v) + \Phi_i^*(N, A, v')$$

for all $i \in N$, and Φ^* satisfies part 1.1 of the linearity axiom. Similarly, for the multiplication by an scalar λ it is the case that $\lambda\phi_k^*(N(A), \hat{v}) = \phi_k^*(N(A), \lambda\hat{v})$ for every $k \in N(A)$ and $\widehat{\lambda v} = \lambda\hat{v}$. Hence,

$$\Phi_i^*(N, A, \lambda v) = \sum_{a\in A}\phi_{i(a)}^*(N(A), \widehat{\lambda v}) = \sum_{a\in A}\phi_{i(a)}^*(N(A), \lambda\hat{v}) =$$
$$\sum_{a\in A}\lambda\phi_{i(a)}^*(N(A), \hat{v}) = \lambda\Phi_i^*(N, A, v)$$

for all $i \in N$, and Φ^* satisfies part 1.2 of the linearity axiom.

(ii) The player anonymity axiom of ϕ^* implies that $\phi_{\sigma k}^*(N(A), \sigma\hat{v}) = \phi_k^*(N(A), \hat{v})$ for any $k \in N(A)$ and for any permutation σ on the set $N(A)$. Take now a permutation σ_N on the set N and denote $\sigma_{N(A)}$ the permutation on the set $N(A)$ that associates player $i(a)$ to $(\sigma_N(i))(a)$, for every $i \in N$, $a \in A$. Consider the game (N, A, v). Then,

$$\widehat{\sigma_N v}(T, Q) = \sum_{a\in A}\sigma_N v\left(T(a); a; Q(b))_{b\in A}\right) = \sum_{a\in A} v(\sigma_N T(a); a; (\sigma_N Q(b))_{b\in A}) =$$
$$\sum_{a\in A} v(\left(\sigma_{N(A)}T\right)(a); a; \left(\left(\sigma_{N(A)}Q\right)(b)\right)_{b\in A}\right) = \hat{v}(\sigma_{N(A)}T, \sigma_{N(A)}Q) = \left(\sigma_{N(A)}\hat{v}\right)(T, Q).$$

Consequently,

$$\Phi_i^*(N, A, \sigma_N v) = \sum_{a\in A}\phi_{i(a)}^*(N(A), \widehat{\sigma_N v}) = \sum_{a\in A}\phi_{i(a)}^*(N(A), \sigma_{N(A)}\hat{v}) =$$
$$\sum_{a\in A}\phi_{\sigma_{N(A)}(i(a))}^*(N(A), \hat{v}) = \sum_{a\in A}\phi_{(\sigma_N(i))(a)}^*(N(A), \hat{v}) = \Phi_{\sigma_N(i)}^*(N, A, v)$$

for each $i \in N$. Hence, Φ^* satisfies the player anonymity axiom.

(iii) We first prove that if $j \in N$ is a dummy player in the game (N, A, v), then all the replicas $j(a)$, for all $a \in A$, are dummy players in $(N(A), \hat{v})$. Consider any $(T, Q) \in ECL(N(A))$ and any (T', Q') obtained from (T, Q) by changing the affiliation of player $j(a)$. For any such (T', Q'), it is always the case that $Q'(b) = Q(b)$ for any $b \neq a$, since we are changing the affiliation of a player that belongs to $N(a)$. There are two possibilities: a) It can be the case that $Q'(a) = Q(a)$. Then,

$$\hat{v}(T', Q') = \sum_{b\in A} v(T'(b); b; (Q'(c))_{c\in A}) = \sum_{b\in A} v(T(b); b; (Q(c))_{c\in A}) = \hat{v}(T, Q).$$

b) Or it can be the case that $Q'(a) \neq Q(a)$ when $j(a)$ changes affiliation. In this case,

$$v(T'(b); b; (Q'(c))_{c\in A}) = v(T(b); b; (Q(c))_{c\in A})$$

for any embedded coalition $(T(b); b; (Q(c))_{c\in A})$ and for all $b \in A$ because $(T'(b); b; (Q'(c))_{c\in A})$ can be deduced from $(T(b); b; (Q(c))_{c\in A})$ by changing the affiliation of the dummy player j within issue a in (N, A, v). Hence again, $\hat{v}(T', Q') = \hat{v}(T, Q)$.

This ends the proof that all the replicas $j(a)$, for all $a \in A$, are dummy players in $(N(A), \hat{v})$.

If ϕ^* satisfies the dummy player axiom, then $\phi^*_{j(a)}(N(A), \hat{v}) = 0$ for all $a \in A$ since $j(a)$ is a dummy player in $(N(A), \hat{v})$. Therefore,

$$\Phi^*_j(N, A, v) = \sum_{a\in A} \phi^*_{j(a)}(N(A), \hat{v}) = 0$$

and Φ^* satisfies the dummy player axiom.

(iv) Consider a dummy player $j \in N$ in the game (N, A, v) and a particular issue $a \in A$. First, since ϕ^* satisfies the strong dummy player property and $j(a)$ is a dummy player in $(N(A), \hat{v})$,

$$\phi^*_k(N(A)\backslash j(a), \hat{v}_{-j(a)}) = \phi^*_k(N(A), \hat{v})$$

for all $k \in N(A)\backslash j(a)$. Second, player $j(b)$, for $b \neq a$, is also a dummy player in the game $(N(A)\backslash j(a), \hat{v}_{-j(a)})$. (If we have two dummy players in any *PFFG*, the second dummy player is still dummy in the game where we have eliminated the first one.) Applying this procedure sequentially to all the issues in A, and denoting $j(A) = \underset{a\in A}{\cup} j(a)$, we have that

$$\phi^*_k(N(A)\backslash j(A), \hat{v}) = \phi^*_k(N(A)\backslash j(A), \hat{v}_{-j(A)})$$

for all $k \in N(A)\backslash j(A)$. Therefore,

$$\Phi^*_i(N\backslash j, A, v_{-j}) = \sum_{a\in A} \phi^*_{i(a)}((N\backslash j)(A), \widehat{v_{-j}}) = \sum_{a\in A} \phi^*_{i(a)}(N(A)\backslash j(A), \hat{v}_{-j(A)}) =$$
$$\sum_{a\in A} \phi^*_{i(a)}(N(A), \hat{v}) = \Phi^*_i(N, A, v)$$

for all $i \in N\backslash j$ and Φ^* satisfies the strong dummy player axiom. ■

Proof of Proposition 2. (i) This property is trivially satisfied.

(ii) If d is a dummy issue in the game (N, A, v), then all the replicas $i(d)$, for any $i \in N$, are dummy players in $(N(A), \hat{v})$, because, by the definition of dummy issue, $\hat{v}(T, Q) = \hat{v}(T', Q')$ for all (T', Q') obtained from (T, Q) by changing the affiliation of player $i(d)$, for any $i \in N$.

Given that ϕ^* satisfies the strong dummy player axiom, then if the n dummy players $i(d)$ are dropped off $N(A)$

$$\phi_k^*(N(A) \backslash (i(d))_{i \in N}, \hat{v}_{-(i(d))_{i \in N}}) = \phi_k^*(N(A), \hat{v}),$$

which implies that for all $i \in N$

$$\Phi_i^*(N, A \backslash d, v_{-d}) = \sum_{a \in A} \phi_{i(a)}^*(N(A \backslash d), \widehat{v_{-d}} = \sum_{a \in A} \phi_{i(a)}^*(N(A) \backslash \{i(d)\}_{i \in N}, \hat{v}_{-(i(d))_{i \in N}}) =$$

$$\sum_{a \in A} \phi_{i(a)}^*(N(A), \hat{v}) = \Phi_i^*(N, A, v)$$

and Φ^* satisfies the dummy issue axiom. ■

Proof of Proposition 3. (i) Consider the game (N, A, v) and, for any $a \in A$, define (N, A, v_a) as

$$\begin{aligned} v_a(S; a; P^A) &\equiv v(S; a; P^A) \text{ for all } (S; a; P^A) \in ECL(N, A) \\ v_a(S; b; P^A) &\equiv 0 \text{ for all } b \in A \backslash a, \ (S; b; P^A) \in ECL(N, A). \end{aligned}$$

It is immediate that $v = \sum_{a \in A} v_a$. The linearity of ϕ^* implies the linearity of Φ^* (Proposition 1); hence

$$\Phi^*(N, A, v) = \sum_{a \in A} \Phi^*(N, A, v_a).$$

Similarly, consider the game (N, A, v_{σ_N}), where σ_N is a permutation of the set of players N. Remember that the function v_{σ_N} is defined as $v_{\sigma_N}(S; a; P^A) \equiv v\left(S; a; \left(P^a, O^{A \backslash a}\right)\right)$ for all $(S; a; P^A) \in ECL(N, A)$, where $O^b = \sigma_N P^b$ for all $b \in A \backslash a$. For any particular $a \in A$, we define $(N, A, (v_{\sigma_N})_a)$ as

$$\begin{aligned} (v_{\sigma_N})_a (S; a; P^A) &\equiv v_{\sigma_N}(S; a; P^A) \text{ for all } (S; a; P^A) \in ECL(N, A) \\ (v_{\sigma_N})_a (S; b; P^A) &\equiv 0 \text{ for all } b \in A \backslash a, \ (S; b; P^A) \in ECL(N, A). \end{aligned}$$

Given that $v_{\sigma_N} = \sum_{a \in A} (v_{\sigma_N})_a$, the linearity of Φ^* implies

$$\Phi^*(N, A, v_{\sigma_N}) = \sum_{a \in A} \Phi^*(N, A, (v_{\sigma_N})_a).$$

We now prove that $\Phi_i^*(N, A, (v_{\sigma_N})_a) = \Phi_i^*(N, A, v_a)$ for all $i \in N$ for whom $\sigma_N(i) = i$, which will prove part (i) of the proposition.

For any $i \in N$,

$$\Phi_i^*(N, A, v_a) = \sum_{b \in A} \phi_{i(b)}^*(N(A), \widehat{v_a})$$

where

$$\widehat{v_a}(T, Q) = \sum_{b \in A} v_a(T(b); b; (Q(c))_{c \in A}) = v_a(T(a); a; (Q(c))_{c \in A})$$

for any $(T, Q) \in ECL(N(A))$, since the other terms in the sum are zero by construction of the function v_a. Also,

$$\Phi_i^*(N, A, (v_{\sigma_N})_a) = \sum_{b \in A} \phi_{i(b)}^*(N(A), \widehat{(v_{\sigma_N})_a})$$

where

$$\widehat{(v_{\sigma_N})_a}(T, Q) = \sum_{b \in A} (v_{\sigma_N})_a (T(b); b; (Q(c))_{c \in A}) = (v_{\sigma_N})_a (T(a); a; (Q(c))_{c \in A})$$

for any $(T, Q) \in ECL(N(A))$. We notice that, by definition of v_{σ_N}, $(v_{\sigma_N})_a (S; a; P^A) = v(S; a; (P^a, O^{A \backslash a}))$ for all $(S; a; P^A) \in ECL(N, A)$, where $O^b = \sigma_N P^b$ for all $b \in A \backslash a$ (and $(v_{\sigma_N})_a (S; b; P^A) = 0$ for all $b \in A \backslash a$, $(S; b; P^A) \in ECL(N, A)$). Since $(v_{\sigma_N})_a$ only permutes the roles of the players involved in issues different from a, $\widehat{(v_{\sigma_N})_a}$ only permutes the roles of the players in each $N(b)$, for all bs different from a. In fact, $\widehat{(v_{\sigma_N})_a} = \sigma_{N(A)} \widehat{v_a}$, where the permutation $\sigma_{N(A)}$ is as follows:

$$\begin{aligned} \sigma_{N(A)}(i(a)) &= i(a) \text{ for all } i \in N \\ \sigma_{N(A)}(i(b)) &= (\sigma_N(i))(b) \text{ for all } i \in N \text{ and all } b \in A \backslash a. \end{aligned}$$

Given that ϕ^* satisfies player anonymity,

$$\phi_{i(a)}^*(N(A), \widehat{(v_{\sigma_N})_a}) = \phi_{i(a)}^*(N(A), \sigma_{N(A)} \widehat{v_a}) = \phi_{i(a)}^*(N(A), \widehat{v_a})$$

for all $i \in N$ and

$$\phi^*_{i(b)}(N(A), \widehat{(v_{\sigma_N})_a}) = \phi^*_{(\sigma_N(i))(b)}(N(A), \sigma_{N(A)}\widehat{v_a}) = \phi^*_{(\sigma_N(i))(b)}(N(A), \widehat{v_a})$$

for all $i \in N$ and all $b \in A \backslash a$. In particular, $\phi^*_{i(b)}(N(A), \widehat{(v_{\sigma_N})_a}) = \phi^*_{i(b)}(N(A), \widehat{v_a})$ for all $i \in N$ for whom $\sigma_N(i) = i$. This implies that $\Phi^*_i(N, A, (v_{\sigma_N})_a) = \Phi^*_i(N, A, v_a)$ for any $i \in N$ for whom $\sigma_N(i) = i$, and the result holds.

(ii) Consider the game (N, A, v), a set M of $a-$externality players and $b \neq a$. We will show that if ϕ^* satisfies linearity and player anonymity in PFFG, then $\Phi^*_i(N, A, v_{M,ab}) = \Phi^*_i(N, A, v)$ for all $i \in N$. Notice that $\Phi^*_i(N, A, v) = \sum\limits_{c \in A} \phi^*_{i(c)}(N(A), \widehat{v})$, where $\widehat{v}(T, Q) = \sum\limits_{c \in A} v(T(c); c; (Q(d))_{d \in A})$ and $\Phi^*_i(N, A, v_{M,ab}) = \sum\limits_{c \in A} \phi^*_{i(c)}(N(A), \widehat{v_{M,ab}})$, where $\widehat{v_{M,ab}}(T, Q) = \sum\limits_{c \in A} v_{M,ab}(T(c); c; (Q(d))_{d \in A})$. We consider the following permutation $\sigma_{N(A)}$ on the set $N(A)$: $\sigma_{N(A)}(i(a)) = i(b)$ and $\sigma_{N(A)}(i(b)) = i(a)$ for all $i \in M$ and $\sigma_{N(A)}(k) = k$ otherwise. Applying the permutation $\sigma_{N(A)}$ to the value function $\widehat{v}$ has the same effect as going from v to $v_{M,ab}$: it moves the roles of players in M from issue a to issue b. Hence, $\sigma_{N(A)}\widehat{v} = \widehat{v_{M,ab}}$.

Given that the value ϕ^* satisfies anonymity, it is the case that

$$\phi^*_{i(c)}(N(A), \widehat{v_{M,ab}}) = \phi^*_{i(c)}(N(A), \sigma_{N(A)}\widehat{v}) = \phi^*_{\sigma_{N(A)}(i(c))}(N(A), \widehat{v}).$$

Given that $\sigma_{N(A)}$ only permutes replicas of the same players (those in M), it is the case that

$$\sum_{c \in A} \phi^*_{i(c)}(N(A), \widehat{v_{M,ab}}) = \sum_{c \in A} \phi^*_{\sigma_{N(A)}(i(c))}(N(A), \widehat{v}) = \sum_{c \in A} \phi^*_{i(c)}(N(A), \widehat{v})$$

(since $\phi^*_{\sigma_{N(A)}(i(a))}(N(A), \widehat{v}) + \phi^*_{\sigma_{N(A)}(i(b))}(N(A), \widehat{v}) = \phi^*_{i(b)}(N(A), \widehat{v}) + \phi^*_{i(a)}(N(A), \widehat{v})$ for $i \in M$). Therefore, $\Phi^*_i(N, A, v_{M,ab}) = \Phi^*_i(N, A, v)$ as we wanted to prove. ■

Proof of Theorem 1. The sufficiency part of the Theorem is a corollary of Propositions 1, 2, and 3. We prove the necessity part through a series of steps. Take any game (N, A, v).

Step 1.- For any $a \in A$, we define the following game (N, A, v_a):

$$\begin{aligned} v_a(S; a; P^A) &\equiv v(S; a; P^A) \text{ for all } (S; a; P^A) \in ECL(N, A) \\ v_a(S; b; P^A) &\equiv 0 \text{ for all } b \in A \backslash a,\ (S; b; P^A) \in ECL(N, A). \end{aligned}$$

That is, the worth of a coalition on issue a in the game v_a is the same as that in v; however, the worth of a coalition on any other issue is zero in game v_a. Note that the organization of the players on issues other than a influences the worth of coalitions in issue a in the game v_a in the same way as it does in v.

It is immediate that

$$v = \sum_{a \in A} v_a.$$

Therefore, if Φ satisfies the axiom of linearity then,

$$\Phi(N, A, v) = \sum_{a \in A} \Phi(N, A, v_a).$$

Step 2.- For each (N, A, v_a), we now define a related game $(N(A), A, w_a)$, which is similar to (N, A, v_a) except that we add $(|A| - 1)n$ dummy players. More precisely, for each $b \in A \setminus a$, let $N(b) = \{i(b) \mid i \in N\}$ be the $b-$replica of N and for convenience, let $N(a) \equiv N$ (i.e., $N(a)$ is the original set of players). Then the set of players in the new game is $N(A) = \cup_{b \in A} N(b)$ with $N(A) \setminus N(a)$ being dummy players. Therefore, $(N(A), A, w_a)$ is defined as follows:[13]

$$w_a(T; a; Q^A) \equiv v_a(T(a); a; Q^A(a))$$

for all $(T; a; Q^A) \in ECL(N(A), A)$ (i.e., for all vector Q^A of $|A|$ partitions of $N(A)$ and any $T \in Q^a$), and

$$w_a(T; b; Q^A) \equiv v_a(T(a); b; Q^A(a)) = 0$$

for all $b \in A \backslash a$ and all $(T; b; Q^A) \in ECL(N(A), A)$.

Given that Φ satisfies the strong dummy player axiom, we have

$$\begin{aligned} \Phi_i(N(A), A, w_a) &= \Phi_i(N, A, v_a) \text{ for all } i \in N(a) = N \\ \Phi_i(N(A), A, w_a) &= 0 \text{ for all } i \in N(A) \backslash N(a). \end{aligned}$$

Step 3.- Next, for each $a \in A$, we define another game $(N(A), A, z_a)$ that is related to $(N(A), A, w_a)$ in the following sense. First, as in $(N(A), A, w_a)$, a coalition of players

[13] As previously done, we denote $T(b) = T \cap N(b)$ for any coalition T of $N(A)$ and $Q(b) = \{T \cap N(b) \mid T \in Q\}$ for any partition Q of $N(A)$, for any $b \in A$.

obtains worth only on issue a. Second, only players in $N(a)$ create worth. Third, the inter-issue externalities in $(N(A), A, z_a)$ are "similar" to those in $(N(A), A, w_a)$; however, there is an important difference: in game $(N(A), A, z_a)$, the externalities originating from each issue $b \in A \backslash a$ are exerted by players in $N(b)$, rather than by players in $N(a)$ as in game $(N(A), A, w_a)$. That is, the game $(N(A), A, z_a)$ is defined as follows:

$$z_a(T; a; Q^A) \equiv w_a(T; a; R^A)$$

for all $(T; a; Q^A) \in ECL(N(A), A)$, where R^A is a vector of $|A|$ partitions of $N(A)$ such that $R^a = Q^a$ and for every $b \in A \backslash a$, R^b is obtained from Q^b by exchanging the memberships of $i(a)$ and $i(b)$ for each $i \in N$,[14] and

$$z_a(T; b; Q^A) \equiv 0$$

for all $b \in A \backslash a$ and all $(T; b; Q^A) \in ECL(N(A), A)$.

Note that $z_a(T; a; Q^A) = v_a\left(T(a); a; \left(\widetilde{Q^b}(b)\right)_{b \in A}\right)$ for all $(T; a; Q^A) \in ECL(N(A), A)$.[15]

We claim that, by issue-externality anonymity axiom,

$$\sum_{b \in A} \Phi_{i(b)}(N(A), A, z_a) = \Phi_i(N, A, w_a) \text{ for all } i \in N = N(a). \tag{4}$$

We prove this claim by decomposing the change from $(N(A), A, w_a)$ to $(N(A), A, z_a)$ in $|N|\,(|A| - 1)$ stages. In each stage, we switch the membership of some $i(a) \in N(a)$ with that $i(b) \in N(b)$ in the partition on some issue $b \in A \backslash a$. In doing so, $i(b)$ takes the role of $i(a)$ in generating externalities from issue b. Note that the identities of the players who create worth (always on issue a) remain the same. Then, by the issue-externality anonymity axiom, the value of every player different from $i(a)$ and $i(b)$ should not change; hence, the sum of the values for players $i(a)$ and $i(b)$ should not change either. Repeating this argument cross issues implies that after $|A| - 1$ stages of switching the membership of $i(a) \in N(a)$ with $i(b) \in N(b)$ for every issue $b \in A \backslash a$, the sum of the values for all replicas of player i remain unchanged while the value of each of the remaining players stays the same throughout these stages. By repeating the above stages for all $i(a) \in N(a)$, we

[14] Thus, $R^b(a) = Q^b(b)$ for all $b \in A \backslash a$.

[15] Recall that Q^b is a partition of $N(A)$ on issue b and $Q^b(b)$ is the partition of $N(b)$ induced by Q^b; $\widetilde{Q^b}(b)$ is obtained from $Q^b(b)$ by replacing each $i(b) \in N(b)$ with i.

complete our transformation from $(N(A), A, w_a)$ to $(N(A), A, z_a)$ and obtain equation (4).

Step 4.- For each $(N(A), A, z_a)$, we now define a related game $(N(A), A, r_a)$ such that all externalities are generated from issue a. Recall that in $(N(A), A, z_a)$, for any $(T; a; Q^A) \in ECL(N(A), A)$, the worth of T depends only on $\left(Q^b(b)\right)_{b\in A}$; moreover, only a coalition of players in $N(a)$ can create worth and it does so only on issue a. In fact, for each $b \in A\backslash a$, $N(b)$ is a set of $b-$externality players in $(N(A), A, z_a)$. We define the game $(N(A), A, r_a)$ by encoding the externalities exerted by $N(b)$ for all $b \in A\backslash b$ in z_a:

$$r_a(T; a; Q^A) \equiv z_a(T; a; R^A)$$

for all $(T; a; Q^A) \in ECL(N(A), A)$ where R^A is a vector of $|A|$ partitions of $N(A)$ such that $R^a = Q^a$ and for every $b \in A\backslash a$, R^b is such that $R^b \cap N(b) = Q^a \cap N(b)$. Thus, r_a can be obtained from z^a from $(|A| - 1)$ steps of transformation, each involving moving the externalities induced by $N(b)$, for a particular $b \in A\backslash a$, from issue b to issue a.

Note that $r_a(T; a; Q^A) = v_a\left(T(a); a; \left(\widetilde{Q^a}(b)\right)_{b\in A}\right)$ for all $(T; a; Q^A) \in ECL(N(A), A)$.

By the issue-externality symmetry axiom,

$$\Phi_k(N(A), A, r_a) = \Phi_k(N, A, w_a) \text{ for all } k \in N(A).$$

We also note that all issues in $A\backslash a$ are dummy issues in $(N(A), A, r_a)$.

Step 5.- Finally, we define game $(N(A), a, s_a)$ by eliminating the set of dummy issues $A\backslash a$ in $(N(A), A, r_a)$, that is,

$$s_a(T; a; Q) \equiv r_a(T; a; Q^A)$$

for any $(T; a; Q) \in ECL(N(A), a)$ and any vector Q^A of $|A|$ partitions of $N(A)$ that satisfies $Q^a = Q$. By the dummy issue axiom, we have

$$\Phi_k(N(A), a, s_a) = \Phi_k(N(A), A, r_a) \text{ for all } k \in N(A).$$

Note that $(N(A), a, s_a)$ is a game with a single issue (a in this case). Therefore, we can consider $(N(A), a, s_a)$ as a PFFG, that we denote $(N(A), \widetilde{s_a})$. Moreover, when it is applied to games with only one issue, the issue symmetry axiom implies that the value Φ depends only on the function that gives the worth of each embedded coalition, not on the identity of the issue itself. Thus, Φ also defines a value for $PFFG$. Let ϕ be this value. Hence,

$$\phi_k(N(A), \widetilde{s_a}) = \Phi_k(N(A), a, s_a) \text{ for all } k \in N(A).$$

Therefore, Steps 1-5 allow us to obtain the following series of equalities for every $i \in N$:

$$\Phi_i(N, A, v) = \sum_{a \in A} \Phi_i(N, A, v_a) = \sum_{a \in A} \Phi_{i(a)}(N(A), A, w_a) =$$
$$\sum_{a \in A}\sum_{b \in A} \Phi_{i(b)}(N(A), A, z_a) = \sum_{a \in A}\sum_{b \in A} \Phi_{i(b)}(N(A), A, r_a) = \sum_{a \in A}\sum_{b \in A} \Phi_{i(b)}(N(A), a, s_a) =$$
$$\sum_{a \in A}\sum_{b \in A} \phi_{i(b)}(N(A), \widetilde{s_a}) = \sum_{b \in A}\sum_{a \in A} \phi_{i(b)}(N(A), \widetilde{s_a}).$$

We now prove that $\hat{v} = \sum_{a \in A} \widetilde{s_a}$. Consider any partition Q of $N(A)$ and any coalition $T \in Q$. By construction,

$$\widetilde{s_a}(T; Q) = s_a(T; a; Q) = r_a(T; a; Q^A),$$

where Q^A is any vector of $|A|$ partitions of $N(A)$ that satisfies $Q^a = Q$. Also,

$$r_a(T; a; Q^A) = v_a\left(T(a); a; \left(\widetilde{Q^a}(b)\right)_{b \in A}\right) = v_a(T(a); a; \left(\widetilde{Q}(b)\right)_{b \in A}) = v(T(a); a; \left(\widetilde{Q}(b)\right)_{b \in A}).$$

Hence,

$$\sum_{a \in A} \widetilde{s_a}(T; Q) = \sum_{a \in A} v(T(a); a; \left(\widetilde{Q}(b)\right)_{b \in A}) = \hat{v}(T, Q).$$

Finally, linearity of Φ implies that the value ϕ is also linear and $\phi_k(N(A), \hat{v}) = \sum_{a \in A} \phi_k(N(A), \widetilde{s_a})$ for all $k \in N(A)$. Therefore,

$$\Phi_i(N, A, v) = \sum_{b \in A} \phi_{i(b)}(N(A), \hat{v})$$

which completes the proof of Theorem 1. ■

Proof of Proposition 4. Take two games (N, A, v) and (N, B, w), with $A \cap B = \varnothing$, and consider a value Φ that satisfies the dummy issue axiom. We add to the first game $|B|$ dummy issues, obtaining the game $(N, A \cup B, v')$ where v' is a characteristic function such that

$$v'(S; a; P^{A \cup B}) = v(S; a; P^A) \text{ for all } a \in A, S \in P^a, P^a \in P^A$$
$$v'(S; b; P^{A \cup B}) = 0 \text{ for all } b \in B, S \in P^b, \text{ and } P^b \in P^B.$$

By the dummy issue property, Φ assigns the same payoff in both games to any player $i \in N$, i.e.,

$$\Phi_i(N, A, v) = \Phi_i(N, A \cup B, v').$$

Similarly, if we add to the game (N, B, w) a set of $|A|$ dummy issues, we obtain the game $(N, A \cup B, w')$ where w' is a characteristic function such that

$$\begin{aligned} w'(S; a; P^{A\cup B}) &= 0 \text{ for all } a \in A, S \in P^a, P^a \in P^A \\ w'(S; b; P^{A\cup B}) &= w(S; b; P^A) \text{ for all } b \in B, S \in P^b, \text{ and } P^b \in P^B. \end{aligned}$$

Again, by the dummy issue axiom, we have

$$\Phi_i(N, B, w) = \Phi_i(N, A \cup B, w'), \text{ for all } i \in N.$$

Since Φ satisfies *linearity*,

$$\Phi_i(N, A \cup B, v') + \Phi_i(N, A \cup B, w') = \Phi_i(N, A \cup B, v' + w').$$

Finally, we notice that the game $(N, A \cup B, v' + w')$ is equivalent to $(N, A \cup B, v' \cup w')$; hence,

$$\Phi_i(N, A, v) + \Phi_i(N, B, w) = \Phi_i(N, A \cup B, v' \cup w')$$

and the independence axiom is satisfied. ■

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