

## **Is the GDP Growth Rate in NIPA a Welfare Measure?**

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# Is the GDP growth rate in NIPA a welfare measure?\*

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**Abstract:** The permanent decline of equipment prices relative to nondurable consumption prices rendered

fixed-base quantity indexes obsolete, because of the well-known substitution bias. National Income and Product Accounts (NIPA) responded by switching to a flexible-base quantity index to measure GDP growth. We argue this is a welfare measure of output growth. In a two-sector endogenous growth model, we use the Bellman equation to explicitly represent preferences on consumption and investment, we apply a Fisher-Shell true quantity index to the this utility representation and show it is equal to the Divisia index, well approximated by the flexible-base quantity index used by NIPA.

**Keywords:** Growth measurement, Quantity indexes, Embodied technical progress, NIPA.

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# 1 Introduction

The Bureau of Economic Analysis (BEA) used to feature in its National Income and Product Accounts (NIPA) a fixed-base quantity index to measure real GDP growth. The observed fast decline of the relative prices of equipment, notably computers and peripheral equipment, has lead BEA to consider alternative measures.<sup>1</sup> The traditional fixed-base Laspeyres quantity index yields a reasonable measurement of real growth provided that relative prices remain stable. If the price of equipment declines, the weight of investment with respect to consumption in the Laspeyres index becomes obsolete quickly enough to have a relevant impact on growth measurement. Trends in relative prices called for flexible-base quantity indexes. Since the early 1990's the NIPA's feature, together with the fixed-base index, a chained-type index built on the Fisher Ideal index. National Accounts in other countries already calculated alternative measures of real growth, like a chained-type index based on the Laspeyres index in the Netherlands and Norway. European Union Member States have also followed BEA: Commission Decision 98/715/EC established 2005 as the beginning of a period in which Member States would progressively adapt their National Accounts. Among these changes stands out the publication of a chained-type index based on the Fisher Ideal index. However, the theoretical legitimation of these measures is still missing. Indeed, modern index number theory is largely built on the idea that index numbers have to reflect the underlying preferences or technology; an important part of the legitimacy of a method to measure real growth stems from the ability of the index to reflect changes in utility in an appropriate and well-defined theoretical framework.

On the other side, growth theory has been reformulated in the late nineties in order to replicate the observed trend in relative prices. Based on Solow (1960), Greenwood et al (1997) propose a simple two-sector optimal growth model where productivity grows faster in the investment sector than in the consumption sector. In this family of models, as in the data, investment grows at a larger rate than consumption, which raises the fundamental problem of measuring output growth. The reason is that two sectors

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<sup>1</sup>Cummins and Violante (2002) contains a thorough review of the evolution of constant-quality prices for equipment from 1947 to 2000 in the US. Since the mid-80's BEA provides with a constant-quality price index for computers and peripherals but historical series first appeared in the seminal contribution of Robert J. Gordon (1990).

growing at different rates render the measurement of real growth a non-trivial exercise. In the one-sector growth model, GDP is simply identified with output in the only sector of the economy. If we have two sectors with two different growth rates, real growth will be somewhere in between these rates, and index number theory can help choosing the right weights.

In this paper, we use the economic theory of index numbers to formulate a true quantity index of output growth and we apply it to the two-sector AK model proposed by Rebelo (1991), which as showed by Felbermayr and Licandro (2005) reproduces the observations referred above. The main contribution of this paper is to show that the chained-type Fisher Ideal index used by NIPA is the welfare-relevant quantity index in this framework. Following Licandro et al (2002), we start from the Bellman equation representation of the planner's problem. Then, to construct a Fisher-Shell true quantity index, we exploit the property that intertemporal preferences may be represented by a utility function over current consumption and current investment. The true quantity index is then proved to be equal to the Divisia index, which in turn is equal to the Fisher Ideal index in this context. The true quantity index is argued to be the welfare-relevant measure of growth in the model economy. Hence, the first contribution of this paper is to show that the Divisia index is the welfare-relevant measurement of growth. In turn, this means that National Accounts real growth rates are welfare-based measurements and, in particular, that the growth rate of investment does contain information relevant to the welfare of the representative individual.

The second contribution of this paper is to clarify an old debate in the growth and growth accounting literatures. The so-called Solow-Jorgenson controversy was revived by the differing views expressed in Hulton (1992) and Greenwood et al (1997). The controversy can be shown to boil down to the issue of the aggregation of consumption and investment when these are measured in different units, and more important, when its relative price has a trend. In our conceptual framework it becomes clear that Greenwood et al (1997) take a more satisfactory path. However, explicitly or implicitly, they — and others following like Oulton (2002)— develop a modern version of the paradigm that consumption, and consequently its growth rate, is the relevant measure of real growth.<sup>2</sup> In this paper, we claim that investment growth, included in the Divisia index,

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<sup>2</sup>Greenwood et al (1997), in facts, is not a normative paper. It does perform the positive exercise of measuring the contribution of embodied technical change to US growth. How-

also matters because it reflects welfare gains from postponed consumption. In a sense, it extends the Weitzman(1976)'s claim that not only consumption but net investment matter for welfare.<sup>3</sup>

Finally, we challenge the view that separates welfare from productivity issues. In this view, output quantity indexes are relevant to productivity measurement while consumer price indexes are relevant for welfare (see e.g. Whelan (2002, p.222)).

Section 2 describes the two-sector, consumption and investment, growth economy and characterize equilibrium. Standard measures of real growth are discussed in Section 3, with particular attention to the important issue of the substitution bias. The main result is shown in Section 4: the ideal-Fisher index used in NIPA is shown to be a welfare measure. Finally, Section 5 discuss the main implications of our results and conclude.

## 2 Two-sector AK model

In this section, we describe a simple version of the two-sector AK model proposed by Rebelo (1991). As shown in Felbermayr and Licandro (2005), it is the simplest endogenous growth model that replicates the observed permanent decline in the price of equipment and the permanent increase in the equipment to output ratio. In this context, it is particularly clear that the aggregation issue is far from trivial since consumption and equipment grow at different rates.

### 2.1 A model of embodied technical progress

The extent to what technical progress affects all production factors (disembodied) or it is incorporated in new machines and therefore embodied in quality-adjusted productive investment (embodied) has been considered a relevant question since the seminal work of Solow (1960). Following this early contribution, embodied technical change (ETC)

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ever, in doing so, they measure GDP and its growth rate in units of consumption. Commins and Violante (2002) generalize the exercise and use standard NIPA methodology to the same objective, finding similar quantitative results.

<sup>3</sup>Secondarily, the exercise entails computing index numbers in model economies. To do this, we develop a general method to compute index numbers—normally applied to actual data, and hence in discrete-time— in continuous-time (see the appendix).

has been usually represented in model economies with a consumption good sector using machines as input and a machines sectors using the consumption good as input. Investment-specific technical progress is interpreted to be embodied in machines, but consumption-specific technical progress is considered to be disembodied. Two recent important contributions follow this tradition. Hulten (1992) in growth accounting and Greenwood et al (1997) in a general equilibrium framework argue that the embodiment hypothesis is a reasonable explanation for the observed decline of equipment prices.

The model in this section is based on Rebelo (1991) and follows Felbermayr and Licandro (2005) closely. It constitutes a simplified version of Greenwood et al (1997) that entails all the characteristics that are relevant to the present discussion. The stock of machines at each instant  $t$  is  $k_t$ , from which  $b_t \leq k_t$  is devoted to the production of the consumption good

$$c_t = b_t^\alpha,$$

where  $\alpha \in (0, 1)$ . The remaining stock  $k_t - b_t$  is devoted to the production of new machines with a linear technology

$$\dot{k}_t = A(k_t - b_t) - \delta k_t,$$

where  $A > 0$ , while  $\delta \in (0, 1)$  is the physical depreciation rate.<sup>4</sup> There is a given initial stock of capital  $k_0 > 0$ . We will write  $i_t = \dot{k}_t$  for net investment. The representative agent has preferences over consumption paths represented by

$$\int_0^\infty e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt$$

where  $\rho > 0$  is the subjective discount rate and  $\sigma \geq 0$  the inverse of the intertemporal elasticity of substitution.

## 2.2 The relative price of equipment

Returns to scale differ between sectors. Since  $\alpha < 1$ , as the stock of capital grows, the equipment sector becomes more productive with respect to the consumption goods

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<sup>4</sup>Observe that  $k_t$  is what in the literature is referred to as effective or quality-adjusted capital. A number of authors discuss that quality-adjusted capital has to be constructed with the physical rather than the economic depreciation rate. This makes the assumption that  $\delta$  is constant consistent with empirical studies. See the discussion in section 3.3 in Cummins and Violante (2002) and the references therein.

sector. This difference in productivity causes the decline of equipment prices relative to consumption goods prices. This difference in returns to scale can be interpreted in terms of the machines sector being more capital intensive than the consumption good sector or, as put forth by Boucekkine et al (2003), as a consequence of strong spillovers among equipment industries.<sup>5</sup>

From the feasibility constraints, we can obtain the competitive equilibrium price of investment in terms of consumption units as the marginal rate of transformation:

$$q_t = -\frac{dc_t}{di_t} = -\frac{dc_t}{db_t} \frac{db_t}{di_t} = \frac{\alpha}{A} b_t^{\alpha-1}.$$

If the stock of machines used in the consumption good sector grows at a constant rate  $g$ , as it will be shown to be the case, the price of equipment decreases at rate  $(\alpha - 1)g < 0$ . Note that this is a model of endogenous growth: there is no exogenous technical change determining this decline of the price of equipment.

## 2.3 Competitive equilibrium

In the absence of market failures, we can represent equilibrium allocations as solutions to the problem of a planner aiming at maximizing utility subject to the technological constraints. The Bellman equation associated to the planner's problem is

$$\rho v(k_t) = \max_{i=A(k-b)-\delta k} \frac{b^{\alpha(1-\sigma)}}{1-\sigma} + v'(k_t)i \quad (1)$$

where the constraint  $c = b^\alpha$  has already been introduced in the objective function. Let  $\mu_t$  be the Lagrange multiplier associated with the technological constraint. The first order conditions<sup>6</sup> are

$$\begin{aligned} \alpha b_t^{\alpha(1-\sigma)-1} - \mu_t A &= 0 \\ v'(k_t) - \mu_t &= 0 \end{aligned}$$

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<sup>5</sup>Cummins and Violante (2002) observe that their measure of investment-specific technical change occurs first in information technology and then accelerates in other industries. They conclude that information technology is a “general purpose” technology, an interpretation that matches well with the spillovers’ interpretation.

<sup>6</sup>This is a concave program. The first order conditions are sufficient if  $\sigma \geq 1$ . When  $0 \leq \sigma < 1$ , we have to require that  $\rho > (1 - \sigma)\alpha(A - \delta)$ . Of course, in general,  $A - \delta > 0$  is necessary for the problem to make sense, and  $A - \delta > \rho$  for positive growth to be optimal. See Felbermayr and Licandro (2005) for the details.

and from the envelope theorem

$$\rho v'(k_t) = v''(k_t)i_t + \mu_t(A - \delta).$$

Since  $\mu_t = v'(k_t)$  we have  $\dot{\mu}_t = v''(k_t)\dot{k}_t$ . Then the envelope theorem equation reads  $-\dot{\mu}_t/\mu_t = A - \delta - \rho$ . Use the first equation to obtain the Euler equation

$$-\frac{\dot{\mu}_t}{\mu_t} = A - \delta - \rho = -(\alpha(1 - \sigma) - 1)\frac{\dot{b}_t}{b_t}.$$

Denote  $g = \dot{b}_t/b_t$  and solve for the growth rate of machines as

$$g = \frac{A - \delta - \rho}{1 - \alpha(1 - \sigma)}.$$

From the feasibility constraints it is clear that  $g$  is the growth rate of capital, and therefore net investment  $i_t$ , and that  $\alpha g$  is the growth rate of consumption. Observe that competitive equilibrium allocations are balanced growth paths as  $g$  is the growth rate of capital stock and investment for all  $t$ .

The competitive equilibrium allocation displays the regularities observed in actual data (see for example Whelan (2003)). Investment grows faster than consumption because  $g > \alpha g$ . The relative price of equipment decreases at rate  $(\alpha - 1)g < 0$ . Consequently, the nominal share of investment in income remains constant. Indeed, from the equations of equilibrium, one can show after some calculations that

$$s_t = \frac{q_t i_t}{m_t} = \frac{q_t i_t}{c_t + q_t i_t} = \frac{\alpha(A - \delta - \rho)}{\rho(1 - \alpha) + \alpha\sigma(A - \delta)} = s$$

for all  $t$ , where  $s$  represents net savings.

Let us take the consumption good as numeraire. Consequently, nominal income is  $m_t = c_t + q_t i_t$ . It is worth noting that this relation is not the resources constraint, since the transformation locus is

$$c_t = \left(k_t - \frac{i_t}{A}\right)^\alpha.$$

Measuring nominal income in consumption units is as arbitrary as measuring it in any other unit. In a later section, we will argue that Greenwood et al (1997) and Oulton (2002) identify production in the consumption good sector with output partly because of a resources constraint that makes this addition look a natural measure of output when in fact it is arbitrary. We also show that, from the perspective of the economic theory of index numbers, it provides with a poor measure of real output growth.

### 3 Actual measures of real growth

The model economy we just described is a world in which investment  $i_t$  grows faster than consumption  $c_t$ , the price of investment relative to consumption  $q_t$  declines, and the nominal shares of consumption and investment expenditures on income remain constant over time, a suitable replication of the facts described by Whelan (2003). Departing from the one-sector growth model raises the issue of aggregating consumption and investment growth, which is particularly relevant since the relative price of investment to consumption shows a trend. In this section, we use the notation of our simple framework to review the methodological changes introduced by the BEA.<sup>7</sup>

#### 3.1 Fixed-base quantity indexes

Traditional measures of real growth stem from fixed-base quantity indexes. The most common among these is the Laspeyres index. Let us choose consumption as the numeraire so that its price is normalized to one while the price of investment in consumption units is  $q_t$ . Let us fix prices at some *base* time  $t$  and then compute the factor of change between  $t$  and  $t + h$  as

$$\Pi_{t+h}^t = \frac{c_{t+h} + q_t i_{t+h}}{c_t + q_t i_t},$$

for all  $h \geq 0$ , where the supraindex  $t$  in  $\Pi$  designate the base time  $t$  and the subindex the current time  $t + h$ . The Laspeyres index  $g_{t+h}^t$  is the instantaneous growth rate of factor  $\Pi_{t+h}^t$  as a function of  $h$  (see the appendix). That is,

$$g_{t+h}^t = \frac{d\Pi_{t+h}^t}{dh} \frac{1}{\Pi_{t+h}^t} = \frac{\dot{c}_{t+h} + q_t \dot{i}_{t+h}}{c_{t+h} + q_t i_{t+h}},$$

which measures the real growth rate at  $t + h$ , for given base time  $t$ . The Laspeyres index is popular because it is conceptually simple.

However, if the relative price of investment permanently declines and real investment permanently grows faster than real consumption, as observed in the data, the Laspeyres

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<sup>7</sup>Young (1992) is a non-technical presentation of the methodological changes introduced in NIPA. Whelan (2002, 2003) is a more detailed guide into the new methods to measure real growth in use at BEA. For economic index number theory see Diewert (1993), Fisher and Shell (1998) and IMF (2004, chapter 17).

index tends to give too much weight to investment as we depart from the base time. In particular, since investment is growing faster than consumption, the Laspeyres growth rate tends to that of investment, therefore overstating real growth. Note that

$$g_{t+h}^t = \frac{c_{t+h}}{c_{t+h} + q_t \dot{i}_{t+h}} \frac{\dot{c}_{t+h}}{c_{t+h}} + \frac{q_t \dot{i}_{t+h}}{c_{t+h} + q_t \dot{i}_{t+h}} \frac{\dot{i}_{t+h}}{i_{t+h}}. \quad (2)$$

It is easy to see that along an equilibrium path with constant income shares, the weight of consumption decreases and the weight of investment increases with  $h$ . This effect is known in the index numbers literature as the *substitution bias*: the demand for goods whose price permanently decline displays faster growth in real terms. Quantity indexes based on past (relatively high) prices overweight these items, overstating the real growth rate. The effect is larger the farther we are from the base year.

The Paasche index uses current prices as a base, and hence tends to understate real growth as we go back in time. The factor of change between  $t - h$  and  $t$  is

$$\Pi_{t-h}^t = \frac{c_t + q_t \dot{i}_t}{c_{t-h} + q_t \dot{i}_{t-h}}$$

for all  $h \geq 0$  and the growth rate measured by the index

$$g_{t-h}^t = \frac{d\Pi_{t-h}^t}{dh} \frac{1}{\Pi_{t-h}^t} = \frac{c_{t-h}}{c_{t-h} + q_t \dot{i}_{t-h}} \frac{\dot{c}_{t-h}}{c_{t-h}} + \frac{q_t \dot{i}_{t-h}}{c_{t-h} + q_t \dot{i}_{t-h}} \frac{\dot{i}_{t-h}}{i_{t-h}}. \quad (3)$$

As  $h$  grows, so  $t - h$  decreases, the weight of consumption increases because  $\dot{i}_{t-h}/c_{t-h}$  decreases.

Fixed-base quantity indexes yield poor measures of real growth when GDP components grow at different rates because of changing relative prices.<sup>8</sup>

### 3.2 Chained-type quantity indexes

The introduction by the BEA of quality corrections in equipment prices in the mid-eighties revealed a persistent declining pattern in the price of equipment relative to the price of nondurable consumption goods. Since then, real investment appears to be

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<sup>8</sup>Updating regularly the base is not a solution because it would imply a permanent revision of past growth performance. It poses the additional problem of multiple real growth measures for each period, each of them affected differently for the substitution bias depending on the associated base period.

growing much faster than real nondurable consumption. In this new scenario, fixed-base quantity indexes face a severe substitution bias problem. For this reason, the BEA moved to a chained-type index based on a Fisher Ideal index computed for contiguous periods.<sup>9</sup>

In continuous time, let the factor of change in the interval  $(t, t + h)$  be the geometric mean of the factors associated to the Laspeyres index with base  $t$  and the Paasche index with base  $t + h$ , that is

$$F_{t,t+h} = \left( \Pi_{t+h}^t \Pi_t^{t+h} \right)^{\frac{1}{2}}.$$

The Fisher Ideal index is the growth rate of factor  $F_{t,t+h}$  as a function of  $h$ . Computing the average compensates the overstatement of the Laspeyres index with the understatement of the Paasche index, thus reducing the impact of the selection bias.

What is a chained-type index? Let time be discrete, say  $t \in \{1, 2, 3, 4, \dots\}$ , and let  $g_t$  be the growth rate measured by a fixed-base index between  $t - 1$  and  $t$ . For example, a Laspeyres index with base period  $t - 1$ . A chained-type index defines the growth between  $t$  and  $t + h$  as<sup>10</sup>

$$1 + g_{t,t+h} = \prod_{i=1}^h (1 + g_{t+i}).$$

It chains the growth rates of a fixed-base index with updated base periods, which reduces the substitution bias because it regularly updates the base. The continuous time version is obtained by letting the length of the period go to zero, in which case

$$g_{t,t+h} = \int_0^h g_{t+x} dx.$$

In continuous time, a chained-type index perfectly counterbalance the substitution bias. It is easy to see in the case of the Fisher Ideal index. When evaluated at  $h = 0$ , as we show in the Appendix, the Fisher Ideal index is equal to the Divisia Index

$$d_t = \frac{c_t}{c_t + q_t i_t} \frac{\dot{c}_t}{c_t} + \frac{q_t i_t}{c_t + q_t i_t} \frac{\dot{i}_t}{i_t},$$

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<sup>9</sup>Diewert (1993) provides a clear explanation of the index suggested by Fisher (1922).

<sup>10</sup>Unlike fixed-base indexes, chain indexes do not have the multiplicative property. In general, the factor  $F_{t,t+2h}$  does not coincide with the chained factor  $F_{t,t+h} \times F_{t+h,t+2h}$ . Just observe that  $t+h$  prices play no role in the calculation of  $F_{t,t+2h}$ . Moreover, both the additive property, that income ratios add up to unity, and the property that income ratios are bounded by unity do not hold. These issues are very well illustrated in Whelan (2002).

which weights consumption and investment growth by their income shares.<sup>11</sup> At any time, the growth rates of consumption and investment are weighted by their current shares, which are independent of any base time. Even if there is a trend in relative prices, inducing the substitution of one good for another, the chained-type index allows weights to change continuously to avoid the emergence of any substitution bias.

In the two-sector AK model, the Divisia index becomes

$$d = (1 - s)\alpha g + sg, \tag{4}$$

where the growth rate of output  $d$  is the weighted average of the growth rates of consumption  $\alpha g$  and investment  $g$ , with constant income shares,  $(1 - s)$  and  $s$ , as weights. The potential substitution bias induced by any fixed-base index is then completely counterbalanced by the chained-type index.

## 4 Real growth in the two-sector AK model

The previous section illustrates how chained-type indexes yield conceptually better measures of real output growth in a scenario in which consumption and investment grow at different rates and there are trends in relative prices. In this section, we go further and show that in the framework of the two-sector AK model, the Divisia index is a true quantity index measuring welfare changes, implying that the NIPA's methodology –a chained-type quantity index based on the Fisher Ideal index– is a true quantity index. In other words, we show that changes in the Divisia index reflect changes in welfare of the infinitely-lived agent.

### 4.1 Instantaneous preferences

In regard to the Bellman equation in (1), we can interpret

$$w_t(c, i) = \frac{c^{1-\sigma}}{1-\sigma} + v'(k_t)i$$

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<sup>11</sup>In continuous-time, as  $h$  goes to zero, the Laspeyres and Paasche indexes tend to each other, implying that chain indexes based on the Laspeyres, Paasche and Fisher Ideal indexes coincide, and all are therefore equal to the Divisia index. In discrete-time, however, only the Fisher Ideal index approximates the Divisia index.

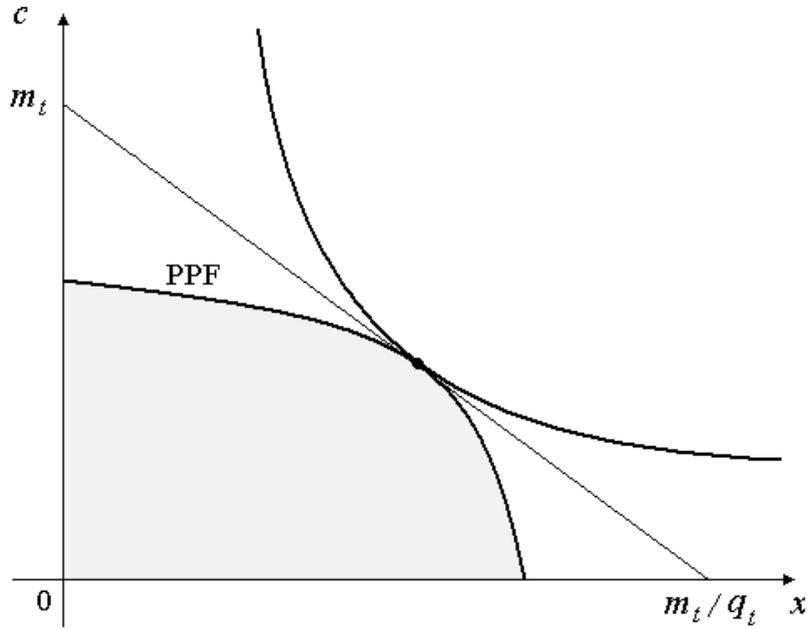


Figure 1: The production possibilities frontier and competitive prices

to represent instantaneous preferences over contemporaneous consumption and investment. To save notation, we write  $w_t(c, i)$  rather than  $w(c, i, k_t)$  but time enters this function only through the stock of capital  $k_t$ .<sup>12</sup> The same comment applies to the indirect utility and expenditure functions below. Recall that nominal income was defined as  $m_t = c_t + q_t i_t$ . Figure 1 depicts the budget constraint jointly with the production possibilities frontier and an indifference curve. The constraints in the maximization problem of the Bellman equation can be replaced by the budget constraint  $c + q_t i \leq m_t$  because the budget line is tangent to the production possibilities frontier locally at the optimum.

## 4.2 Fisher-Shell true quantity index

Since the utility function that summarizes preferences in instant  $t$  is time dependent, the natural choice to measure real growth is the true quantity index introduced by Fisher

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<sup>12</sup>The planner solves a standar recursive program in which the state variable summarizes at each instant all information from the past that could be relevant for the choice today. For a brief exposition of recursive techniques in continuous-time see Obstfeld (1992).

and Shell (1971). Define the indirect utility function as

$$u_t(m_t, q_t) = \max_{c+q_t i \leq m_t} w_t(c, i)$$

and the associate expenditure function

$$e_t(u_t, q_t) = \min_{w_t(c, i) \geq u_t} c + q_t i.$$

Since utility comparisons must be done using the same preference map, the Fisher-Shell true quantity index fixes both prices and preferences. In particular, it compares income today  $m_t$  with the hypothetical level of income  $\hat{m}_{t+h}$  that would be necessary to attain the level of utility associated with tomorrow's income and prices  $m_{t+h}, q_{t+h}$  with today's prices  $q_t$  and today's preferences as evaluated by  $e_t, u_t$ . Denote this artificial level of income as

$$\hat{m}_{t+h} = e_t[u_t(m_{t+h}, q_{t+h}), q_t].$$

The procedure is depicted in Figure 2. The preference map corresponds to preferences of instant  $t$  as represented by  $w_t$ . Point A is the current situation in instant  $t$ . Point B is the choice of instant  $t$  preferences when we face instant  $t+h$  prices  $q_{t+h}$  and income  $m_{t+h}$ . Point C represents the choice that maintains such level of utility but with instant  $t$  prices  $q_t$ . In the end, we compare two levels of income that correspond to the same price vector so it is clear that we are extracting price changes. In this particular case, the true quantity index is just reflecting the fact that the true output deflator is dropping with the price of equipment, that is to say that income in real terms is growing more than  $m_{t+h}/m_t$ .

The instantaneous Fisher-Shell real growth rate is

$$f_t = \left. \frac{d}{dh} \frac{\hat{m}_{t+h}}{m_t} \right|_{h=0} = \frac{1}{m_t} \left. \frac{d\hat{m}_{t+h}}{dh} \right|_{h=0}, \quad (5)$$

that is, the instantaneous growth rate of the factor defined above as  $h$  gets small.<sup>13</sup> To compute this index note that

$$\left. \frac{d\hat{m}_{t+h}}{dh} \right|_{h=0} = e_{1,t}[u_t(m_t, q_t), q_t](u_{1,t}(m_t, q_t)\dot{m}_t + u_{2,t}(m_t, q_t)\dot{q}_t)$$

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<sup>13</sup>In continuous-time, it does not make any difference whether we define the true quantity index taking past, like we do, or future prices and preferences as given.

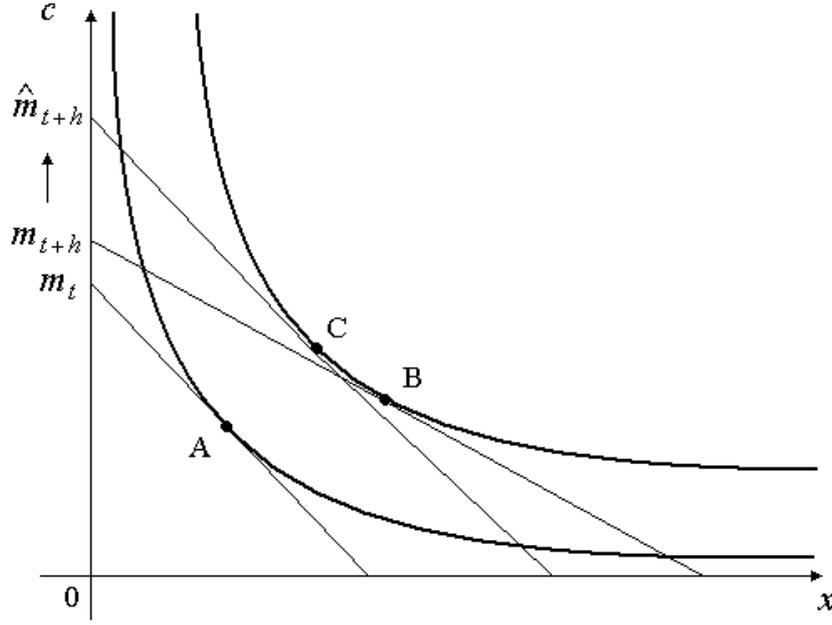


Figure 2: Fisher-Shell true quantity index

where the subscript denotes the partial derivative with respect to the corresponding argument. To obtain an expression for all these derivatives let us go back to the definitions above. Let  $\mu$  be the Lagrange multiplier of the maximization problem in the definition of the indirect utility function. We have, from the first order conditions, that  $w_{1,t}(c_t, i_t) = \mu$ , so that

$$\begin{aligned} \frac{du_t(m_t, q_t)}{dm_t} &= \mu = w_{1,t}(c_t, i_t) \\ \frac{du_t(m_t, q_t)}{dq_t} &= -\mu i_t = -w_{1,t}(c_t, i_t) i_t. \end{aligned}$$

Let  $\lambda$  be the Lagrange multiplier of the minimization problem in the definition of the expenditure function. From the first order conditions  $w_{1,t}(c_t, i_t) = -\lambda^{-1}$ , and hence

$$\frac{de_t(u_t, q_t)}{du_t} = -\lambda = \frac{1}{w_{1,t}(c_t, i_t)}.$$

We conclude that

$$f_t = \frac{1}{m_t} \frac{1}{w_{1,t}(c_t, i_t)} (w_{1,t}(c_t, i_t) \dot{m}_t - w_{1,t}(c_t, i_t) i_t \dot{q}_t) = \frac{\dot{m}_t - i_t q_t \dot{q}_t}{m_t} = \frac{\dot{m}_t}{m_t} - \frac{q_t i_t \dot{q}_t}{m_t q_t}.$$

Differentiate  $m_t = c_t + q_t i_t$  with respect to time and note that  $q_t i_t / m_t = s$  to write

$$\frac{\dot{m}_t}{m_t} = (1 - s) \frac{\dot{c}_t}{c_t} + s \frac{\dot{i}_t}{i_t} + s \frac{\dot{q}_t}{q_t}$$

and then

$$f_t = \frac{\dot{m}_t}{m_t} - s \frac{\dot{q}_t}{q_t} = (1-s) \frac{\dot{c}_t}{c_t} + s \frac{\dot{i}_t}{i_t} = (1-s)\alpha g + sg$$

for all  $t$ , but this is the expression of the Divisia index in (4) so that  $f_t = d$  for all  $t$ .<sup>14</sup>

The interpretation is straightforward. It is clear that  $f$  is a measure of real growth since it is constructed as the growth rate of nominal income subtracting pure price changes, in this case the change of the relative price of investment  $q_t$ . The index only keeps changes in quantities. It is also clear that it is a true index because it is constructed from the representative agent's preferences.<sup>15</sup>

## 5 Discussion

If current and future consumption is all that matters for utility, in principle, it is very appealing to think that the growth rate of consumption should summarize all what the agents care about. However, we have shown above that investment also matters. In this section, we explain why the growth rate of investment does play a role in the true quantity index.

*What does the Fisher-Shell index measure?* The value function  $v(k)$  in (1) measures the utility that the representative individual derives from her consumption flow given the stock of assets  $k$ . However, as is well known, the value function is one among many possible representations of individual preferences. We have then to adopt a sensible norm to measure changes in welfare. In order to make welfare comparisons, index number

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<sup>14</sup>This equivalence would come as no surprise to index number theorists. The Fisher Ideal index is known to approximate in general some sort of true quantity index because both are bounded from above and below by the Laspeyres and Paasche indexes respectively. In continuous-time, these indexes tend to each other as the time interval  $h$  tends to zero. Further, in general, the Divisia index coincides with the Fisher Ideal index if the growth rates of consumption and investment are constant.

<sup>15</sup>At this point it may be worth clarifying that it is not GDP but NDP what matters for welfare, as initially stated by Weitzman (1976), and then is the contribution of net investment  $i_t$  what the true quantity index is taking into account. Depreciated capital is a lost resource that does not contribute to welfare. It is in this sense that some authors claim that NDP is relevant for welfare and GDP for productivity (see the discussion in Oulton (2002)). In the two-sector AK model, however, net and gross investment grow at the same rate.

theory uses current income as a norm, such as in the Fisher-Shell index defined in section 4.2. Moreover, the Bellman equation (1) says that the return to the value of current assets has to be equal to the sum of the value of current consumption and current net investment, which is measured by current net income when the Fisher-Shell index is adopted. This is the way we interpret Weitzman (1976)'s statement that net output is a welfare measure: for a given norm, it does measure in utility terms the return to assets, which is proportional to individual welfare as represented by  $v(k)$ . The Fisher-Shell index goes one step farther by measuring changes in output consistently with individual preferences; when taking current income as a norm, the Fisher-Shell index measures in welfare terms the increase in assets returns and, since assets returns are proportional to  $v(k)$ , it does measure gains in welfare from  $t$  to  $t + dt$ .

*Why does net investment also matters?* The classical paper referenced above, Weitzman (1976), makes a compelling case in favor of including net investment in any measurement of the income level. The reason cannot be more intuitive: for a given stock of real assets, a welfare assessment of output has to complement current consumption with changes in future consumption, in turn summarized in current net investment. Less obvious, however, is how to extend this argument to growth rates. The reason is that, in principle, the growth rate of consumption along a balanced growth path is giving us already information about future consumption. However, the propensity to invest and the growth rate of investment contain useful information concerning the value of future consumption too. To see this, note first that the Fisher-Shell-Divisia index can be written as:

$$d = (A - \delta)s.$$

That is, the index reflects that potential maximum growth  $A - \delta$  will be exploited to the extent that agents care for future consumption as represented by the propensity to invest as measured by the net saving rate  $s$ .

This interpretation can be further supported by a simple example. The basic intuition is that the same growth rate for consumption can stem from different parameters' profiles that, eventually, reflect different preferences. Consider the economy described above and a second economy such that  $\rho > \tilde{\rho}$  and  $\delta < \tilde{\delta}$ . Suppose further that  $\delta + \rho = \tilde{\delta} + \tilde{\rho}$ . The rest of parameters are common to both economies. In regard to the equilibrium expressions for  $s$  and  $g$ , we have  $g = \tilde{g}$  and therefore  $\alpha g = \alpha \tilde{g}$  so that consumption grows at the same rate in both economies. Since optimal paths are balanced growth paths, we can easily

derive the functional form of the value function associated to the planner's problem

$$v(k) = \frac{\theta^{\alpha(1-\sigma)-1} k^{\alpha(1-\sigma)}}{A(1-\sigma)} \quad (6)$$

where  $\theta = b_t/k_t$  can be expressed in terms of parameters as

$$\theta = \frac{\rho - \alpha(1-\sigma)(A-\delta)}{A(1-\alpha(1-\sigma))}.$$

Hence, decreasing  $\rho$  in the same amount as  $\delta$  increases the numerator because  $-1 + \alpha(1-\sigma) < 0$ , while this same expression, negative, is the power of  $\theta$  in (6).

In short, the growth rate of consumption is not a good measure of real growth because it is unable to reflect the welfare differences between these two economies: the second economy weights more future consumption so that values more than the first the same growth rate of consumption. The Fisher-Shell-Divisia index does reflect the differential in welfare. Since  $g = \tilde{g}$ , if there is a difference between  $d$  and  $\tilde{d}$  it has to be because of differences in the propensity to invest. Check that

$$s = \frac{\alpha(A-\delta-\rho)}{\rho(1-\alpha) + \alpha\sigma(A-\delta)} = \frac{\alpha(A-\tilde{\delta}-\tilde{\rho})}{\rho(1-\alpha) + \alpha\sigma(A-\delta)} < \frac{\alpha(A-\tilde{\delta}-\tilde{\rho})}{\tilde{\rho}(1-\alpha) + \alpha\sigma(A-\tilde{\delta})} = \tilde{s}$$

where the equality follows from the imposition that  $\delta + \rho = \tilde{\delta} + \tilde{\rho}$  and the inequality from rising  $\delta$  and decreasing  $\rho$  in the denominator. But then

$$d = (\alpha + s(1-\alpha))g < (\alpha + \tilde{s}(1-\alpha))g = \tilde{d}$$

so that  $d < \tilde{d}$  is reflecting the higher valuation in the second economy. This example shows that the Divisia index is a better representation of preference than the consumption growth rate.

Finally, notice that in our simple model economy preferences and technology are such that competitive equilibria are balanced growth paths. But one could conceive other economies in which a positive shock to the ETP term would depress consumption in the short-term reflecting a substitution of current for future consumption. In which case, focusing on the growth rate of consumption in the short-term can be even more misleading.

*What are the implications for growth accounting?* In terms of model representations of actual economies, the introduction of more than one sector with different growth rates raises practical and conceptual issues. From the practical point of view, we ought to be

careful when calibrating a model to match real data —what is the model equivalent of real GDP growth in National Accounts? To measure the contribution of ETC to growth, Hulten (1992) first measures growth (his equation (7)) following Jorgenson (1966) suggesting a raw addition of consumption and investment units, calling the outcome quality-adjusted output. Using our notation, this strategy amounts to  $c_t + x_t/q_t = c_t + i_t$ . Greenwood et al (1997) note that, in their setting, adding consumption and effective investment turns the economy into a standard Solow (1960) growth model with no embodied technical change.<sup>16</sup> Furthermore, the addition  $c_t + i_t$  implies adding goods measured in different units, mixing potatoes with machines. Greenwood et al (1997) correctly state that output has to be measured in a common unit and they adopt the nondurable good as measure. For this purpose, investment has to be deflated by its relative price, in our notation  $c_t + q_t i_t = m_t$ . However, doing so they effectively identify output growth with growth of output in the non-durable sector. Hence, their measurement of GDP growth is the growth rate of consumption.<sup>17</sup> Oulton (2004) generalizes the argument and suggests that GDP components have to be deflated by the consumption price index in order to measure growth. But this is indeed what Greenwood et al (1997) suggest when they identify non-durable production with real output and the real growth rate with the growth rate of consumption, i.e.  $\alpha g < d$ .<sup>18</sup> What this paper aims to clarify is that the issue is not the units used to measure real output levels but the choice of the right index of real output growth. We conclude that the “true” contribution of ETC to output growth, reflecting welfare changes, has to be measured using NIPA’s methodology as in Cummins and Violante (2002).

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<sup>16</sup>See Hercowitz (1998) for a review of the Solow-Jorgenson controversy.

<sup>17</sup>In their setting, this choice looks somewhat natural because the machines’ sector uses as input the consumption good so that  $m_t = c_t + q_t i_t$  is total output in the non-durables sector.

<sup>18</sup>The controversy between Hulten (1992) and Greenwood et al (1997) has another important element not related to the measurement of growth. In particular, it is the short-term growth-accounting approach of the former in contrast with the long-term equilibrium growth rate of the latter. In Durán et al (2006) (in Spanish) we argue that these approaches are complementary rather than opposite.

## 6 Conclusions

In the framework of the two-sector AK model, which replicates the well-know stylized facts that investment growth faster than consumption and the relative price of investment permanently declines, we show that a Fisher-Shell true quantity index is equal to the Divisia index. This allows us to claim that the chained-type index used by the BEA to compute real GDP growth is a welfare measure.

This approach may be of great relevance for the recent debate on the use on index number theory to rationalize the use of the Penn World Tables (see Neary (2004) and van Veelen and van der Weide (2008)).

## Appendix

### Quantity indexes in continuous-time

Suppose we have a general definition of an index  $\Gamma_{t,t+h}$  interpreted as a gross rate (or factor) of growth between  $t$  and  $t+h$ . We can define the instantaneous growth rate of the index in instant  $t$  as

$$\gamma_t = \left. \frac{d\Gamma_{t,t+h}}{dh} \frac{1}{\Gamma_{t,t+h}} \right|_{h=0}. \quad (7)$$

The intuition of (7) is clear if one observes that the growth rate of a factor of change of a continuous-time variable is equal to the growth rate of the variable itself. Let  $x_t$  be a continuous-time variable and fix some reference point in time  $t$ . The growth rate in instant  $t+h$  can be seen as the growth rate of  $\Gamma_{t,t+h} = x_{t+h}/x_t$  because

$$\gamma_{t,t+h} = \frac{d\Gamma_{t,t+h}}{dh} \frac{1}{\Gamma_{t,t+h}} = \frac{\dot{x}_{t+h}x_t}{x_t^2} \frac{1}{\frac{x_{t+h}}{x_t}} = \frac{\dot{x}_{t+h}}{x_{t+h}}$$

and of course the growth rate in instant  $t$  is just

$$\gamma_t = \left. \frac{d\Gamma_{t,t+h}}{dh} \frac{1}{\Gamma_{t,t+h}} \right|_{h=0} = \left. \frac{\dot{x}_{t+h}}{x_{t+h}} \right|_{h=0} = \frac{\dot{x}_t}{x_t}.$$

This way of defining the instantaneous growth rate may look odd but it may be useful in those cases in which we have an index like  $\Gamma_{t,t+h}$  but no clear variable giving rise to this index like  $x_t$  in this example. The Fisher Ideal index is one of these cases.

Using the notation introduced in section 2, the starting point is some nominal aggregate  $c_t + q_t \dot{i}_t$ . Fixed-base quantity indexes use instant  $\ell$  prices as weights and compute an index that is equal to a factor of growth

$$\Pi_t^\ell = \frac{c_t + q_t \dot{i}_t}{c_\ell + q_\ell \dot{i}_\ell}.$$

When we measure real growth over an interval  $[0, T]$  the index  $\Pi_t^\ell$  is the Laspeyres index when  $\ell = 0$  and the Paasche index when  $\ell = T$ . The Fisher Ideal index between  $t$  and  $t + h$  is defined as

$$F_{t,t+h} = \left( \Pi_{t+h}^t \Pi_t^{t+h} \right)^{\frac{1}{2}}. \quad (8)$$

There is a clear parallelism between  $F_{t,t+h}$  and  $\Gamma_{t,t+h}$  but there is no counterpart for the  $x_t$  variable above. Note that the definition of the index itself requires to set some reference point in time  $t$  and a second reference  $t + h$ . Further, the non-linearity of expression (8) does not make it easy to turn some discrete-version of the growth rate into a derivative that could yield an instantaneous growth rate in an intuitive way.

Instead, we can apply the definition (7) above and define

$$g_t = \left. \frac{dF_{t,t+h}}{dh} \frac{1}{F_{t,t+h}} \right|_{h=0}.$$

With this definition at hand, it is straightforward to check that the continuous-time equivalent of the Fisher Ideal index is in fact equal to the Divisia index. We have

$$\begin{aligned} \left. \frac{dF_{t,t+h}}{dh} \frac{1}{F_{t,t+h}} \right|_{h=0} &= \frac{1}{2} \left( \Pi_{t+h}^t \Pi_t^{t+h} \right)^{-1} \left( \frac{d\Pi_{t+h}^t}{dh} \Pi_t^{t+h} + \Pi_{t+h}^t \frac{d\Pi_t^{t+h}}{dh} \right) \\ &= \frac{1}{2} \left( \frac{d\Pi_{t+h}^t}{dh} \frac{1}{\Pi_{t+h}^t} + \frac{d\Pi_t^{t+h}}{dh} \frac{1}{\Pi_t^{t+h}} \right). \end{aligned}$$

Then note that

$$\left. \frac{d\Pi_{t+h}^t}{dh} \frac{1}{\Pi_{t+h}^t} \right|_{h=0} = \left. \frac{\dot{c}_{t+h} + q_t \dot{i}_{t+h}}{c_t + q_t \dot{i}_t} \right|_{h=0} = \frac{\dot{c}_t + q_t \dot{i}_t}{c_t + q_t \dot{i}_t}$$

while

$$\begin{aligned} \left. \frac{d\Pi_t^{t+h}}{dh} \frac{1}{\Pi_t^{t+h}} \right|_{h=0} &= \left. \frac{(\dot{c}_{t+h} + \dot{q}_{t+h} \dot{i}_{t+h} + q_{t+h} \dot{i}_{t+h})(c_t + q_{t+h} \dot{i}_t) - (c_{t+h} + q_{t+h} \dot{i}_{t+h}) \dot{q}_{t+h} \dot{i}_t}{(c_t + q_{t+h} \dot{i}_t)^2} \right|_{h=0} \end{aligned}$$

and therefore

$$\left. \frac{d\Pi_t^{t+h}}{dh} \frac{1}{\Pi_t^{t+h}} \right|_{h=0} = \frac{\dot{c}_t + \dot{q}_t \dot{i}_t + q_t \dot{i}_t - \dot{q}_t \dot{i}_t}{c_t + q_t \dot{i}_t} = \frac{\dot{c}_t + q_t \dot{i}_t}{c_t + q_t \dot{i}_t}.$$

We conclude that

$$g_t = \frac{dF_{t,t+h}}{dh} \frac{1}{F_{t,t+h}} \Big|_{h=0} = \frac{1}{2} \left( \frac{\dot{c}_t + q_t \dot{i}_t}{c_t + q_t i_t} + \frac{\dot{c}_t + q_t \dot{i}_t}{c_t + q_t i_t} \right) = \frac{\dot{c}_t + q_t \dot{i}_t}{c_t + q_t i_t} \\ = \frac{c_t}{c_t + q_t i_t} \frac{\dot{c}_t}{c_t} + \frac{q_t i_t}{c_t + q_t i_t} \frac{\dot{i}_t}{i_t},$$

that is, the Divisia index.

The definition above (7) is also useful applied to the Fisher-Shell quantity index since we have a well-defined index  $\hat{m}_{t+h}/m_t$  but it is not clear who would play the role of  $x_t$  in this case. Section 4 takes this viewpoint and defines the growth rate as in expression (5) in page 12.

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