Conflict-free and Pareto-optimal Allocations in the One-sided Assignment Game: A Solution Concept Weaker than the Core

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Conflict-free and Pareto-optimal allocations in the one-sided assignment game: A solution concept weaker than the core*

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Abstract

In the one-sided assignment game any two agents can form a partnership and decide how to share the surplus created. Thus, in this market, an outcome involves a matching and a vector of payoffs. Contrary to the two-sided assignment game, stable outcomes often fail to exist in the one-sided assignment game. We introduce the idea of conflict-free outcomes: they are individually rational outcomes where no matched agent can form a blocking pair with any other agent, neither matched nor unmatched. We propose the set of Pareto-optimal (PO) conflict-free outcomes, which is the set of the maximal elements of the set of conflict-free outcomes, as a natural solution concept for this game. We prove several properties of conflict-free outcomes and PO conflict-free outcomes. In particular, we show that each element in the set of PO conflict-free payoffs provides the maximum surplus out of the set of conflict-free payoffs, the set is always non-empty and it coincides with the core when the core is non-empty. We further support the set of PO conflict-free outcomes as a natural solution concept by suggesting an idealized partnership formation process that leads to these outcomes. In this process, partnerships are formed sequentially under the premise of optimal behavior and two agents only reach an agreement if both believe that more favorable terms will not be obtained in any future negotiations.

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1 Introduction

Interactions among people, firms, and many other agents, often take place in terms of two-agent partnerships. A seller and a buyer meet to realize a transaction that is profitable for both; a firm and a worker sign a contract that benefits both; two firms establish an R&D collaboration agreement; or two roommates agree to share the cost of an apartment. Some of these partnerships take place between pairs of agents from two clearly distinct sets: there is a set of buyers and a set of sellers, as there is a set of firms and a set of workers. A buyer, for instance, is either matched with a seller or he/she does not buy, but he is not interested in forming a partnership with another buyer. In other environments, pairs are made between agents who all belong to the same set: a set of innovative firms or a set of tenants. A firm, for instance, may be matched to any other firm or it can do R&D on its own.

Matching models, pioneered by Koopmans and Beckmann (1957), Gale and Shapley (1962), and Shapley and Shubik (1972), provide an excellent framework to study interactions when the agents belong to two different subpopulations. The success of these models is due, at least partially, to the very nice properties of the solution concepts used. In particular, in the models of two-sided two-agent partnerships, both in the discrete environment (the marriage model) and the continuous environment (the assignment game) the core always exists, it coincides with the set of pairwise-stable allocations and it has appealing properties.

In our paper, we study environments where agents from a single population, and not necessarily from two distinct sets, match and where the agents endogenously decide not only on their partners but also on the sharing of the surplus created by the partnerships. Thus, an outcome of our model involves both a matching (that is, a partition of the population in either pairs of agents or singletons) and a vector of payoffs (that is, a sharing of the joint surplus for any two-agent partnership). We will refer to this game as the "one-sided assignment game."[1] Despite their interest, these environments have received very little attention in the economics literature. One important reason is that, contrary to the two-sided assignment game, stable allocations often fail to exist.[2]

When it is not empty, the set of stable payoffs in the one-sided assignment game has similar properties to that in the two-sided assignment game. Indeed, we prove that the set of stable payoffs coincides with the set of pairwise stable payoffs and with the core of the game; hence, we will refer to this solution concept as corewise-stability. Moreover, as Sotomayor (2005a and 2009a) and Talman and Yang (2011) show, any optimal matching[3] is compatible with any corewise-stable payoff and every matching in a corewise-stable outcome is optimal.

[1] It is called "the partnership formation problem" in Talman and Yang (2011) and Andersson et al. (2014), "the TU roommate game" in Eriksson and Karlander (2000), and simply "the roommate problem" in Chiappori et al. (2014).

[2] Gale and Shapley (1962) show that stable matchings may also not exist in the one-sided discrete model, that is, when utility is not transferable. The existence problem for that model was called by these authors "the roommate problem."

[3] A matching is optimal if it maximizes the total payoff in the set of feasible matchings.

While corewise-stability is a very appealing solution concept, the fact that the set of corewise-stable allocations may be empty is problematic because it does not provide a prediction for many environments. The main purpose of this paper is to introduce the idea of a "conflict-free outcome" in the one-sided assignment game, to propose the "set of Pareto-optimal (PO) conflict-free outcomes" as a natural solution concept for this game, and to analyze the properties of this set.

Conflict-free outcomes share properties of core outcomes but not necessarily all of them. Conflict-free payoffs (the vectors of payoffs in conflict-free outcomes) are individually rational. Moreover, no matched agent in a conflict-free outcome can form a blocking pair with any other agent, neither matched nor unmatched. Hence, conflict-free outcomes are somehow "internally stable." However, they might not be "externality stable," in the sense that there may be a pair of agents not involved in any partnership that could have an incentive to deviate. As is clear from the definition, all core outcomes are conflict-free outcomes. However, the set of conflict-free outcomes is always non-empty; hence, some conflict-free payoffs do not belong to the core.

A PO conflict-free payoff is identified with a maximal element (under the partial order relation induced from that of $\mathbb{R}^n$) of the set of conflict-free outcomes, which is a non-empty compact set. Then, a first import property of our proposed solution concept is that PO conflict-free outcomes always exist; hence, the set of PO conflict-free payoffs is always non-empty. Additionally, we prove that the set of PO conflict-free outcomes always has a structure similar to that of the set of core outcomes. Indeed, every PO conflict-free outcome provides the maximum total surplus among all conflict-free outcomes; and no other conflict-free outcome can achieve this level of total surplus. Thus, the matchings that are compatible with PO conflict-free outcomes are "quasi-optimal." Moreover, similar as the corewise-stable outcomes, each quasi-optimal matching is compatible with each PO conflict-free payoff. Therefore, the set of PO conflict-free outcomes is the Cartesian product of the set of quasi-optimal matchings and the set of PO conflict-free payoffs.

The last main property satisfied by the set of PO conflict-free outcomes that we mention here is that the set of PO conflict-free payoffs coincides with the core, when the core is non-empty. Therefore, our solution concept proposes the core when it is not empty and, when the core is empty, it recommends a set of payoffs that is, in some sense, as stable as possible and that satisfies properties that are similar to the core.

We also show that every conflict-free outcome can be "extended" to a PO conflict-free outcome, in the sense that each agent matched in the conflict-free outcome will keep his/her payoff in the PO conflict-free outcome. In fact, we further support the set of PO conflict-free outcomes as a natural solution concept for the one-sided assignment game by suggesting an idealized dynamic environment that leads to these outcomes. In this process, partnerships are formed sequentially under the premise of optimal behavior and two agents only reach an agreement if both believe that more favorable terms will not be obtained in any future negotiations. That is, once a transaction is done at a given stage, the agents involved will keep their payoffs at the subsequent stages.

Starting from the conflict-free outcome where everybody stands alone, we can construct a finite sequence of conflict-free outcomes that gradually increases cooperation through Pareto improvements, still staying within the set of conflict-free outcomes. The process stops when no new transaction is able to benefit the agents involved—in which case we reach the core—or until the outcome cannot be conflict-free anymore. The final conflict-free outcome of the sequence has the property that improving the payoff of some agent through a conflict-free outcome makes another agent worse off. That is, any final outcome of this sequence is Pareto-optimal conflict-free.

Finally, the properties that the set of PO conflict-free payoffs always exists and that it coincides with the core when the core is not empty allow us to provide necessary and sufficient conditions for the core to be non-empty based on the examination of PO conflict-free payoffs.

As stated above, few papers have studied the one-sided assignment game. Necessary and sufficient conditions for the existence of the core using linear programming are obtained by Talman and Yang (2011). Erikson and Karlander (2000) use graph theory to provide a characterization of the core, and Klaus and Nichifor (2010) provide some properties of this set, when it is not empty. Chiappori et al. (2014) show that stable matchings exist when the economy is replicated an even number of times by "cloning" each individual. Finally, Andersson et al. (2014) propose a dynamic competitive adjustment process that either leads to a stable outcome or disproves the existence of stable outcomes.

The idea of a conflict-free outcome is similar to that of a "simple outcome," which was introduced in Sotomayor (1996 and 2005b) to provide very short proofs, which only use elementary combinatorial arguments, of the existence of stable outcomes (or the core) in the marriage model and in the housing markets with strict or non-strict preferences (Shapley and Scarf, 1974). Still in environments without transfers, the notion of a simple outcome was used in Sotomayor (1999) for a discrete many-to-many matching model with substitutable and not-necessarily strict preferences, in Sotomayor (2004) where an implementation mechanism for the discrete many-to-many matching model is provided, in Sotomayor (2011) to characterize the set of Pareto-stable matchings in the marriage market and (if the set of stable matchings is not empty) in the discrete roommate model, and in Wu and Roth (2018) for the college admission model. An adaptation of the concept of simple matching was used in Sotomayor (2000) for a unified two-sided matching mode, due to Eriksson and Karlander (2000), which includes the marriage and the assignment model, and in Sotomayor (2018) for the two-sided assignment game of Shapley and Shubik (1972).

The rest of the paper is organized as follows. Section 2 introduces the framework and states some preliminary results for the core. Section 3 introduces the conflict-free outcomes and provides some of their properties. Section 4 analyzes the set of PO conflict-free outcomes and presents our results on this set. In section 5, we prove the links between the set of PO conflict-free outcomes and the set of corewise-stable outcomes. The final remarks are given in section 6.

2 Framework and preliminaries

2.1 The framework

The description of the one-sided assignment game follows the one given in Roth and Sotomayor (1990) for the case with two sides, with the appropriate adaptations.

There is a finite set of players, $N = \{1, 2, ..., n\}$. Associated with each partnership $\{i, j\}$ there is a non-negative real number $a_{\{i,j\}}$ which will be denoted a_{ij}. The number a_{ij} represents the surplus that players i and j generate if they form a partnership.

We can represent the environment as a game in coalitional function form (N, v) with side payments determined by (N, a). In this game, the worth $v(i, j)$[4] of a two-player coalition $\{i, j\}$ is given by a_{ij}. We will define $v(i) \equiv a_{ii} \equiv 0$ for all $i \in N$. The worth of larger coalitions is entirely determined by the worth of the pairwise combinations that the coalition members can form. That is, $v(S) = max\{v(i_1, j_1) + v(i_2, j_2) + ... + v(i_k, j_k)\}$ for arbitrary coalitions S, where the maximum is taken over all sets $\{i_1, j_1\}, ..., \{i_k, j_k\}$ of two-player disjoint coalitions in S.[5]

Thus, the rules of the game are that any pair of agents $\{i, j\}$ can together obtain a_{ij}, and any larger coalition is valuable only insofar as it can organize itself into such pairs. The members of any coalition may divide their collective worth among themselves in any way they like.

We might think of the two-sided "Assignment Game" of Shapley and Shubik (1972) as a particular case of our model. In the assignment game, there are two disjoint sets P and Q and a pair of players can generate a surplus only if each belongs to a different set. Thus, our model corresponds to an assignment game when $N = P \cup Q$, $P \cap Q = \varnothing$, and $v(S) = 0$ if S contains only agents of P or only agents of Q.

We will represent the set of partnerships that are formed through a matching:

Definition 1 *A **feasible matching** x is a partition of N, where the partition sets are either pairs $\{i, j\}$ or singletons $\{i\}$. If $\{i, j\} \in x$ we can write $x(i) = j$ and we refer to $x(i)$ as the partner of i at x. If $\{i\} \in x$ we can write $x(i) = i$ and we say that i is **unmatched** at x.*

We will use the notation $\sum_A$ to denote the sum over all elements of A. Let x be a feasible matching. If $R \subseteq N$, we denote $\boldsymbol{x(R) \equiv \{j; x(i) = j}$ **for some** $\boldsymbol{i \in R\}}$. If $x(R) = R$, we denote by $x|_R$ the partition of R where the partition sets belong to x. Therefore, $v(R) \geq \sum_{x|_R} a_{ij}$ for all feasible matchings x.

Definition 2 *The feasible matching x is optimal if, for all feasible matching x', $\sum_x a_{ij} \geq \sum_{x'} a_{ij}$.*

The set of optimal matchings is always non-empty, since there is a finite number of matchings. Under Definition 2 and since $v(N) \geq \sum_x a_{ij}$ for all feasible matchings x, it follows that the matching $\boldsymbol{x}$ **is optimal if and only if** $\boldsymbol{\sum_x a_{ij} = v(N)}$.

The players' benefit in the game will be represented by a vector of payoffs:

[4] For notational convenience, we write $v(i, j)$ rather than $v(\{i, j\})$.

[5] k is an integer number that does not exceed the integer part of $|S|/2$.

Definition 3 *The vector u, with $u \in \mathbb{R}^n$, is called the* ***payoff****. The payoff u is* ***pairwise-feasible*** *for (N, a) if there is a feasible matching x such that*

$$u_i + u_j = a_{ij} \text{ if } x(i) = j \text{ and } u_i = 0 \text{ if } x(i) = i.$$

In this case, we say that (u, x) is a ***pairwise-feasible outcome*** *and x is compatible with u.*

Definition 4 *The payoff u is feasible for (N, a) if $\sum_N u_i \leq v(N)$.*

Remark 1 *Given a coalition R, the definition of v implies that there is some feasible matching x such that $x(R) = R$ and $\sum_{x|_R} a_{ij} = v(R)$. Furthermore, $v(R) \geq \sum_{x'|_R} a_{ij}$ for all feasible matchings x' such that $x'(R) = R$. Then, it follows from Definition 3 that* ***$\sum_R u_i \leq v(R)$ for all $R \subseteq N$ and for all pairwise-feasible outcomes (u, x) with $x(R) = R$. In particular, $\sum_N u_i \leq v(N)$****. Therefore,* ***every pairwise-feasible payoff is feasible****.* ■

The natural solution concept is that of stability (the general definition of stability is given in Sotomayor, 2009b). For the one-sided assignment game, stability is equivalent to the concept of **pairwise-stability**.

Definition 5 *The pairwise-feasible payoff u is* ***pairwise-stable*** *if*

(i) $u_i \geq 0$ *for all* $i \in N$ *and*

(ii) $u_i + u_j \geq a_{ij}$ *for all* $\{i, j\} \subseteq N$.

If x is compatible with u we say that (u, x) is a pairwise-stable outcome.

Condition (i) (individual rationality) means that in a pairwise-stable situation a player always has the option of remaining unmatched. Condition (ii) ensures the stability of the payoff distribution: If it is not satisfied for some agents i and j then it would pay for them to break up their present partnership(s) and form a new one together, as this would give them each a higher payoff. In this case, we say that $\{i, j\}$ blocks u.

We now define the *core* of (N, a), which we denote by C:

Definition 6 *We say that $u \in C$ if $\sum_N u_i = v(N)$ and $\sum_S u_i \geq v(S)$ for all $S \subseteq N$.*

The following example shows that the core of this model may be empty.

Example 1 *Consider $N = \{1, 2, 3\}$ and $a_{ij} = 1$ for all $\{i, j\} \subseteq N$. For every feasible payoff u there exist two players i and j such that $u_i + u_j < 1$. Hence, the core of this game is empty.*

Definition 7 *Let (u, x) be a pairwise-feasible outcome. Let $R \subseteq N$. We say that* ***R is a stable coalition for (u, x)*** *if (a) $x(R) = R$, (b) $u_i + u_{x(i)} = a_{ix(i)}$ for all $i \in R$ and (c) $u_i + u_j \geq a_{ij}$ for all $\{i, j\} \subseteq R$.*

Remark 2 *Notice that if R is stable for (u, x) it must be the case that $\sum_R u_i \geq v(R)$, according to Definition 7. On the other hand, $\sum_R u_i \leq v(R)$, as stated in Remark 1. Therefore,* ***$\sum_R u_i = v(R)$****.*
■

2.2 Preliminary results for the core

In our environment, the concepts of stability and the core are equivalent, as established in the following proposition.

Proposition 1 *The set of pairwise-stable payoffs coincides with the core of* (N, a)*.*

Proof. Suppose u is a pairwise-stable payoff. Then, u is feasible and so

$$\sum_N u_i \leq v(N) \tag{1}$$

according to Remark 1. Moreover, consider any coalition S and let y be a feasible matching such that $y(S) = S$ and $v(S) = \sum_y a_{ij}$. The pairwise-stability of u implies that $u_i + u_{y(i)} \geq a_{iy(i)}$ for all $i \in S$, so

$$\sum_S u_i \geq v(S) \text{ for all coalition } S. \tag{2}$$

Under (1) and (2) it follows that $\sum_N u_i = v(N)$ and $\sum_S u_i \geq v(S)$ for all $S \subseteq N$, so u is in the core.

Now, suppose u is in the core. Definition 6 implies that $u_i + u_j \geq v(i,j) = a_{ij}$ for every coalition $\{i,j\}$ and $u_i \geq v(i) = 0$ for all $i \in N$, so u does not have any blocking pair and is individually rational. To see that u is pairwise-feasible, let x be a feasible matching such that $v(N) = \sum_x a_{ij}$. Use that $\sum_N u_i = v(N)$ and $u_i + u_{x(i)} \geq a_{ix(i)}$ for all $i \in N$, to get that $\sum_N u_i = \sum_x a_{ij} \leq \sum_{i \leq x(i)}(u_i + u_{x(i)}) = \sum_N u_i$, so the inequality cannot be strict, which implies $u_i + u_{x(i)} = a_{ix(i)}$ for all $i \in N$. Since $u_i \geq 0$, it follows that $u_i = 0$ if $x(i) = i$. Hence, u is pairwise-stable and the proof is complete. ■

In what follows, given its equivalence with the core concept, the concept of pairwise-stability will be called **corewise-stability**.

The following two propositions, proven by Sotomayor (2005a, 2009a) and Talman and Yang (2011), make clear why, similarly to the two-sided assignment game and in contrast to the discrete version (the roommate-problem), we can concentrate on the payoffs to the agents rather than on the underlying matching. Indeed, the propositions show that the set of corewise-stable outcomes is the Cartesian product of the set of corewise-stable payoffs and the set of optimal matchings. We state the results without the proofs.

Proposition 2 *If x is an optimal matching then it is compatible with any corewise-stable payoff u.*

A consequence of Proposition 2 is that, similarly to the two-sided assignment game, every unmatched player in a corewise-stable outcome has a zero payoff at any corewise-stable outcome in the one-sided assignment game. Corollary 1 states this result.

Corollary 1 *Let x be an optimal matching. If i is unmatched at x then $u_i = 0$ for all corewise-stable payoffs u.*[6]

Proof. Let $u \in C$. Under Proposition 2, u is compatible with x, so $u_i = 0$ by the pairwise-feasibility of u. ■

Proposition 3 *If (u, x) is a corewise-stable outcome then x is an optimal matching.*

3 Conflict-free outcomes

In this section, we introduce a key solution concept for the theory developed in this paper: a conflict-free outcome. Conflict-free outcomes satisfy properties similar to, but weaker than, stable outcomes.

Definition 8 *The outcome (u, x) is conflict-free if it is pairwise-feasible, individually rational and no blocking pair $\{i, j\}$ exists where either i or j are matched at x. A matching x is conflict-free if there is some payoff u such that the outcome (u, x) is conflict-free. The payoff u is a* ***conflict-free payoff*** *if there is some matching x such that (u, x) is a conflict-free outcome.*

Clearly, every corewise-stable outcome is conflict-free. However, conflict-free outcomes are not necessarily stable. For instance, the outcome where every player is unmatched and obtains a payoff of 0 is conflict-free, but it is not corewise-stable in any game where at least one partnership creates a positive surplus. Moreover, this example of a conflict-free outcome allows us to state that **the set of conflict-free outcomes is always non-empty**.

To discuss the difference between corewise-stable and conflict-free outcomes, consider a pairwise-feasible and individually rational outcome (u, x). We denote by $T(x)$ the set of all players who are matched at x, and by $U(x)$ the set of players who are unmatched at x. That is,

$$T(x) \equiv \{j \in N; x(j) \neq j\} \text{ and } U(x) \equiv N \backslash T(x).$$

The outcome (u, x) is conflict-free if and only if no player in $T(x)$ can form a blocking pair neither with another player in $T(x)$ (which implies that the coalition $T(x)$ is a stable coalition for (u, x)) nor with any player outside $T(x)$. In this sense, we could say that a conflict-free outcome is "internally stable." This is why we say that $T(x)$ is the stable active coalition for (u, x). However, to be corewise-stable the outcome also needs to be "externally stable," in the sense that no pair of players outside $T(x)$ can block the outcome either. A conflict-free outcome might not be "externally stable."

We denote by S the set of conflict-free payoffs:

$$S \equiv \{u \in \mathbb{R}^n; u \text{ is conflict-free}\}.$$

[6] This result was proved in Demange and Gale (1985) for a two-sided matching market where the utilities are continuous, so it applies to the two-sided assignment game of Shapley and Shubik (1972).

Also, given any conflict-free matching x, denote:

$$S(x) \equiv \{u \in S; u \text{ is compatible with } x\}.$$

Remark 3 *Notice that if (u,x) is conflict-free then the set of players $T(x)$ who are matched at x and all the subsets $R \subseteq T(x)$ such that $x(R) = R$ are stable coalitions for (u,x). Therefore, by Remark 2, $\sum_{T(x)} u_i = v(T(x))$, and for such coalitions R, we also have $\sum_R u_i = v(R)$.* ■

Our next results highlight relationships between conflict-free matchings and optimal matchings, as well as between conflict-free outcomes and corewise-stable outcomes. Lemma 1, which is interesting by itself, will help to prove these relationships.

Lemma 1 *Let (u,x) a conflict-free outcome. Then, $v(T(x)) + v(U(x)) = v(N)$.*

Proof. Let y be an optimal matching. Then, $v(N) = \sum_y a_{ij}$. Set

$$\alpha \equiv \{\{i,j\} \in y; \{i,j\} \cap T(x) \neq \varnothing \text{ and } \{i,j\} \cap U(x) \neq \varnothing\},$$

$$\beta \equiv \{\{i,j\} \in y; \{i,j\} \subseteq T(x)\} \text{ and}$$

$$\gamma \equiv \{\{i,j\} \in y; \{i,j\} \cap T(x) = \varnothing\}.$$

Also, denote

$$\alpha_x \equiv \{i \in T(x); \{i,j\} \in \alpha \text{ for some } j\}$$

and

$$R \text{ is a set of pairs } \{i,j\} \subseteq U(x) \text{ such that } v(U(x)) \equiv \sum_R a_{ij},$$

$$R' \equiv U(x) \backslash \cup_\gamma \{i,j\} \text{ and}$$

$$R'' \text{ is a set of pairs } \{i,j\} \subseteq T(x) \text{ such that } v(T(x)) \equiv \sum_{R''} a_{ij}.$$

Then, $\sum_{T(x)} u_i = \sum_{\alpha_x} u_i + \sum_\beta a_{ij} \geq \sum_\alpha a_{ij} + \sum_\beta a_{ij}$, where the inequality is due to the fact that $(u,x) \in S$, $i \in T(x)$, and $u_{x(i)} = 0$ for all $i \in \alpha_x$, and so $u_i = u_i + u_{x(i)} \geq a_{ix(i)}$ for all $i \in \alpha_x$.

Also, $R' \cup \gamma$ is a partition of $U(x)$, so $v(U(x)) = \sum_R a_{ij} \geq \sum_\gamma a_{ij} + \sum_{R'} a_{ii} = \sum_\gamma a_{ij}$. Then,

$$v(N) = \sum_y a_{ij} = \left(\sum_\alpha a_{ij} + \sum_\beta a_{ij}\right) + \sum_\gamma a_{ij} \leq \sum_{T(x)} u_i + v(U(x)) = v(T(x)) + v(U(x)),$$

where in the last equality it was used that $T(x)$ is a stable coalition. Therefore,

$$v(N) \leq v(T(x)) + v(U(x). \tag{3}$$

On the other hand, y is an optimal matching, so $v(N) = \sum_y a_{ij} \geq \sum_{R \cup R''} a_{ij} = v(T(x)) + v(U(x)$. Then,

$$v(N) \geq v(T(x)) + v(U(x). \tag{4}$$

Hence, $v(T(x)) + v(U(x) = v(N)$ and the proof is complete. ■

Proposition 4 states that a conflict-free matching may not be optimal but it is always part of an optimal matching.

Proposition 4 *Let (u, x) be a conflict-free outcome. Then, the set of active partnerships of x is part of an optimal matching.*

Proof. The proof is immediate after the fact that $v(T(x))+v(U(x)) = v(N)$ becuase of Lemma 1, $N = T(x) \cup U(x)$ and $T(x) \cap U(x) = \varnothing$. ■

Remark 4 *We notice that if the conflict-free outcome (u, x) is unstable then $v(U(x)) > 0$. In fact, let $\{j, k\}$ be a blocking pair. Then $\{j, k\} \subseteq U(x)$, so*

$$0 = \sum_{U(x)} u_i = (u_j + u_k) + \sum_{U(x)\setminus\{j,k\}} u_i < a_{jk} + \sum_{U(x)\setminus\{j,k\}} a_{ii} \leq v(U(x)).$$

Therefore, $v(U(x)) > 0$. ■

Proposition 5 states a result complementary to Proposition 4: the only conflict-free payoffs compatible with an optimal matching are the corewise-stable payoffs.

Proposition 5 *Let (u, x) be a conflict-free outcome. Suppose x is optimal. Then $u \in C$.*

Proof. Denote R a set of pairs $\{i, j\} \subseteq T(x)$ such that $v(T(x)) \equiv \sum_R a_{ij}$. Since x is optimal, then

$$v(N) = \sum_x a_{ij} = \sum_{x|_{T(x)}} a_{ij} + \sum_{x|_{U(x)}} a_{ij} = \sum_{x|_{T(x)}} a_{ij} = v(T(x)),$$

where the last equality follows from Proposition 4. Then, under Lemma 1, we have that $v(U(x)) = 0 = \sum_{U(x)} u_i$ so, as stated in Remark 4, there is no blocking pair in $U(x)$, which implies that (u, x) is corewise-stable. Hence, we have completed the proof. ■

Proposition 5 leads to the following corollary.

Corollary 2 *A conflict-free outcome (u, x) is corewise-stable if and only if $\sum_N u_i = v(N)$.*

Proof. Consider the conflict-free outcome (u, x). If it is corewise-stable then $\sum_N u_i = v(N)$ according to Definition 6. On the hand, if $\sum_N u_i = v(N)$ then the matching x is necessarily optimal, so u is a corewise-stable payoff as stated in Proposition 5. ■

We now show that the set of conflict-free payoffs is a compact set of $\mathbb{R}^n$, a property that we will use in the next section.

Proposition 6 *The set of conflict-free payoffs S is a compact set of $\mathbb{R}^n$.*

Proof. The set S is bounded because $0 \leq u_j \leq v(N)$ for all $j \in N$ and for all conflict-free payoffs u. To prove that it is also closed, take any sequence $(u^t)_{t=1,2,...}$ of conflict-free payoffs, with $u^t \to u$ when t tends to infinity. Since the set of matchings is finite, there is some matching x which is compatible with infinitely many terms of the sequence $(u^t)_{t=1,2,...}$. Denote $(v^t)_{t=1,2,...}$ this subsequence. Then, if $x(j) = k$, $u_j + u_k = lim_{t\to\infty}(v_j^t + v_k^t) = lim_{t\to\infty} a_{jk} = a_{jk}$. Similarly, if $x(j) = j$ then $u_j = lim_{t\to\infty} v_j^t = 0$. Thus, x is compatible with u, so (u, x) is feasible. We claim that if j is matched at x then j is not part of a blocking pair of (u,x). In fact, $u_j + u_k = lim_{t\to\infty}(v_j^t + v_k^t) \geq lim_{t\to\infty} a_{jk} = a_{jk}$ for any $k \in N\backslash\{j\}$, where the inequality holds because (v^t, x) is a conflict-free outcome for all t. Therefore, (u,x) is a conflict-free outcome, so u is a conflict-free payoff. Hence, the set of conflict-free payoffs is bounded and closed, so it is compact. ■

Remark 5 *If x is a conflict-free matching then there exists some conflict-free payoff which is compatible with x, so $S(x)$ is non-empty. On the other hand, the arguments used in the proof of Proposition 6 also hold if we require that $(u_t)_{t=1,2,\ldots}$ is a sequence of conflict-free payoffs in $S(x)$. Therefore, the set* ***$S(x)$ is a non-empty and compact set of*** *$\mathbb{R}^n$.*

Next, we prove an important *Decomposition Lemma* for the set of conflict-free outcomes which has similarities with other decomposition lemmas in matching models (see, for instance, Gale and Sotomayor, 1985). The lemma states that, for any two conflict-free outcomes, a player who is matched at both outcomes and obtains a higher payoff in the first is necessarily matched, at both outcomes, to a player who obtains a higher payoff in the second.

Lemma 2 *Let (u,x) and (w,y) be conflict-free outcomes. Let $M_u \equiv \{j \in T(y); u_j > w_j\}$ and $M_w \equiv \{j \in T(x); w_j > u_j\}$. Then $x(M_u) = y(M_u) = M_w$ and $x(M_w) = y(M_w) = M_u$.*[7]

Proof. We first prove that $x(M_u) \subseteq M_w$. Take $j \in M_u$; then j is matched under x since $u_j > w_j \geq 0$. We show by contradiction that $k \equiv x(j)$ is in M_w. Suppose $k \notin M_w$, then

$$a_{jk} = u_j + u_k > w_j + w_k$$

which implies that (j,k) blocks (w,y). However, $j \in M_u$ so it is matched at y, which contradicts that (w,y) is conflict-free.

A similar argument leads to $y(M_w) \subseteq Mu$.

Moreover, $x(M_u) \subseteq M_w$ implies $M_u \subseteq x(M_w)$ and $y(M_w) \subseteq M_u$ implies $M_w \subseteq y(M_u)$. Since all the players in M_u and in M_w are matched at x and y, it follows that $|M_u| = |x(M_u)|$, $|M_w| = |y(M_w)|$, $|y(M_u)| = |M_u|$ and $|y(M_u)| = |M_u|$. Therefore,

$$|M_u| = |x(M_u)| \leq |M_w| = |y(M_w)| \leq |M_u|$$

and

$$|M_w| \leq |y(M_u)| = |M_u| \leq |x(M_w)| = |M_w|,$$

which imply $x(M_u) = M_w$, $y(M_w) = M_u$, $y(M_u) = M_w$, and $x(M_w) = M_u$. ■

As can be seen in the proof of Lemma 2, $u_j > 0$ for all $j \in M_u$ and $w_j > 0$ for all $j \in M_w$. Therefore, we can write $M_u = \{j \in T(x) \cap T(y); u_j > w_j\}$ and $M_w \equiv \{j \in T(x) \cap T(y); w_j > u_j\}$.

[7] The decomposition lemma applies, in particular, to core outcomes. Then, an immediate consequence of the lemma is a polarization of interests between the partners along the core: If (u,x) and (w,y) are corewise-stable outcomes, j is matched to k under x or under y, and $u_j > w_j$, then $w_k > u_k$. This is because both payoffs are compatible with the same optimal matching; therefore, if j is matched to k under (u,x) then j is also matched to k under (w,x), so Lemma 2 applies.

Finally, we introduce the idea of an extension of a conflict-free outcome, which will be useful in the next section. In words, a feasible outcome (w, z) extends the conflict-free outcome (u, x) if all the players in the stable active coalition of (u, x) keep their payoff but some players who were unmatched in (u, x) obtain a positive payoff (hence, they are matched) in (w, z).

Definition 9 *Let (u, x) be a conflict-free outcome. We say that the feasible outcome* ***(w, z) extends (u, x)*** *if $w_j > u_j$ for some $j \notin T(x)$ and $w_j = u_j$ for al $j \in T(x)$. If (w, z) is conflict-free (respectively, corewise-stable) then (w, z) is said to be a conflict-free (respectively, corewise-stable) extension of (u, x).*

Sometimes, we will refer to a conflict-free outcome that does not have any extension as a *non-extendable outcome.*

Proposition 7 states that any Pareto improvement of a conflict-free outcome through another conflict-free outcome is necessarily an extension of that outcome.

Proposition 7 *Let (u, x) and (w, y) be conflict-free outcomes. Suppose $w > u$.*[8] *Then (w, y) is a conflict-free extension of (u, x).*

Proof. To show that (w, y) is a conflict-free extension of (u, x), we need to prove that $j \notin T(x)$ for all j such that $w_j > u_j$ (see Definition 9). Consider a player j such that $w_j > u_j$ and suppose, by contradiction, that $j \in T(x)$. Given that $w_j > 0$, we have that $j \in T(y)$. Then, $j \in T(x) \cap T(y)$ and so $j \in M_w$. Denote $k \equiv y(j)$. Lemma 2 implies that $k \in M_u$. Therefore, $u_k > w_k$, which contradicts the assumption that $w > u$. Hence, (w, y) extends (u, x). ■

4 Pareto-optimal conflict-free outcomes

Of particular interest for our analysis is the set of the conflict-free outcomes that are not dominated, via coalition N, by any other conflict-free outcome. This section introduces the set of Pareto-optimal conflict-free outcomes and provides important properties of this set. To introduce the set, let us first formally define the notion of Pareto optimality.

Definition 10 *Let A be a set of payoffs. The payoff u is* ***Pareto-optimal (PO) in A****(or among all payoffs in A) if it belongs to A and there is no payoff w in A such that $w > u$.*

If u is Pareto-optimal in A and x is compatible with u, we say that (u, x) is a Pareto-optimal outcome in A.

The case in which A corresponds to the set of conflict-free payoffs, that is, $A = S$, plays an important role in our theory.

[8] Given two vectors $w, v \in \mathbb{R}^n$, we will denote $w > u$ if $w_j \geq u_j$ for all players $j \in N$ and $w_j > u_j$ for at least one player $j \in N$.

Definition 11 *The payoff u is a* ***PO conflict-free payoff*** *if it is a conflict-free payoff and it is PO in the set of conflict-free payoffs. The outcome (u, x) is a* ***PO conflict-free outcome*** *if (u, x) is a conflict-free outcome and u is a PO conflict-free payoff.*

The set of PO conflict-free payoffs will be denoted by S^*:

$$S^* \equiv \{u \in S; u \text{ is Pareto optimal in } S\}.$$

We notice that, given Definition 9, every PO conflict-free outcome is non-extendable.

Similarly, if u is Pareto optimal in A and A is the set of individually rational and feasible payoffs we will refer to u as a PO feasible payoff.

Remark 6 *It follows from Definition 10 that an individually rational and feasible payoff u is PO feasible if and only if $\sum_N u_i = v(N)$. Thus, every corewise-stable payoff is PO feasible. However, the Pareto optimality of a payoff is not enough to guarantee its corewise-stability. For instance, in Example 1, the payoff $u = (1, 0, 0)$ is not in the core but it is PO feasible, since $\sum_N u_i = v(N)$.* ■

The first property that we will state concerning the set of PO conflict-free payoffs is that, similar to the set of conflict-free payoffs, the set of PO conflict-free payoffs is compact. The proof requires a previous lemma.

Lemma 3 *Let A be a non-empty and compact set of $\mathbb{R}^n$, ordered with the partial order relation $\geq$ induced by $\mathbb{R}^n$. Then, the set of maximal elements of A with respect to $\geq$ is a non-empty and compact set of $\mathbb{R}^n$.*

Proof. Denote $A^* \equiv \{u \in A; u$ is a maximal element of $A\}$. It is known that every non-empty, compact and partially ordered set has a maximal element, so $A^* \neq \varnothing$. The set A^* is clearly bounded, since A is bounded. To see that A^* is closed, take any sequence of vectors $(u^t)_{t=1,2,\ldots}$, with $u^t \in A^*$ for all t, which converges to some vector u. Suppose, by way of contradiction, that $u \notin A^*$. Then, there exists some vector $w \in A$ such that $w > u$. If this is the case, there is some neighborhood V of the vector u and some integer k such that $u^t \in V$ for all $t \geq k$ and $w > u'$ for all $u' \in V$. In particular, $w > u^k$, which contradicts the assumption that $u^k \in A^*$. Hence, A^* is a compact set of $\mathbb{R}^n$. ■

Proposition 8 *The set of PO conflict-free payoffs S^* is a non-empty and compact set of $\mathbb{R}^n$.*

Proof. According to Proposition 6, S is compact and non-empty. Moreover, S is an ordered set by the partial order relation $\geq$ induced by $\mathbb{R}^n$. Then, Lemma 3 applies and so S^*, the set of maximal elements of S, is a non-empty and compact set of $\mathbb{R}^n$. ■

PO conflict-free payoffs are, by definition, undominated in the set S. Next result shows that every conflict-free payoff which is not a PO conflict-free payoff is necessarily dominated for some PO conflict-free payoff.

Proposition 9 *Let u be a conflict-free payoff which is not PO conflict-free, that is, $u \in S \backslash S^*$. Then there is some PO conflict-free payoff w_u such that $w_u > u$.*

Proof. Suppose, by way of contradiction, that there is no payoff w in S^* such that $w > u$. Since $u \notin S^*$, there is some $w^1 \in S$ such that $w^1 > u$. Then, by contradiction, $w^1 \notin S^*$, so there is some $w^2 \in S$ such that $w^2 > w^1 > u$. Again, w^2 cannot be in S^*. By repeating this procedure, we obtain an infinite sequence $(w^t)_{t=1,2,...}$ of conflict-free payoffs with distinct terms. On the other hand, there is a finite number of conflict-free matchings, so there is some conflict-free matching x which is compatible with infinitely many terms of the sequence. Denote $(v^t)_{t=1,2,...}$ that subsequence. All the members of the subsequence are in S so $\sum_{T(x)} v_j^t = v(T(x))$ for all $t = 1, 2, \ldots$. However, $v^1 < v^2$, so $v(T(x)) = \sum_{T(x)} v_j^1 = \sum_N v_j^1 < \sum_N v_j^2 = \sum_{T(x)} v_j^2 = v(T(x))$, which is an absurd. Hence, there is some $w_u \in S^*$ such that $w_u > u$. ■

Propositions 8 and 9 allow us to establish an interesting corollary: there is some payoff in S^* that dominates every conflict-free payoff outside S^*.

Corollary 3 *There is some PO conflict-free payoff $w^* \in S^*$ such that $\sum_N w_j^* > \sum_N u_j$ for all $u \in S \backslash S^*$.*

Proof. By Proposition 8 we have that S^* is a non-empty and compact set of $\mathbb{R}^n$. Since every continuous function defined in a compact set has a maximum in this set, there is some $w^* \in S^*$ such that $\sum_N w_j^* \geq \sum_N w_j$ for all $w \in S^*$. Now use Proposition 9 to get that $\sum_N w_j^* > \sum_N u_j$ for all $u \in S \backslash S^*$. ■

Proposition 9, together with Proposition 7, also implies that not only are the PO conflict-free outcomes non-extendable but they are the only non-extendable outcomes. This result is stated in Corollary 4.

Corollary 4 *The set of PO conflict-free outcomes equals the set of non-extendable outcomes.*

Proof. Let $(u, x) \in S^*$. Then (u, x) cannot have any conflict-free extension, as stated in Definition 9. The other direction is immediate from propositions 9 and 7. ■

The next property that we prove is that all PO conflict-free outcomes are equally efficient, in the sense that the players' total payoff is the same in every PO conflict-free payoff. This property will be proven in Proposition 10, which requires two previous lemmas.

Lemma 4 *Let (u, x) and (w, y) be PO conflict-free outcomes. Let $j^* \in T(x) \backslash T(y)$ with $a_{j^* x(j^*)} > 0$. Then $x(j^*) \in T(y)$.*

Proof. Suppose, by way of contradiction, that $x(j^*) \in T(x) \backslash T(y)$. Denote $A \equiv \{t \in T(x) \backslash T(y); x(t) \in T(x) \backslash T(y)\}$. We have that $j^* \in A$, so $A \neq \varnothing$.

We first show that

$$w_{q^*} + u_{t^*} = u_{t^*} < a_{q*t^*} \text{ for some } t^* \in A \text{ and } q^* \in N \backslash (T(y) \cup A). \quad (5)$$

Suppose, by way of contradiction, that there are no such players t^* and q^*. Then, either $N\backslash(T(y) \cup A) = \varnothing$ or (given that $w_q = 0$ for all $q \notin T(y)$) $u_t \geq a_{qt}$ for all $q \in N\backslash(T(y) \cup A)$ and $t \in A$. In any case, we can construct the outcome (w', y') as follows: the matching y' agrees with x on A and it agrees with y on $N\backslash A$, hence, $T(w') = T(y) \cup A$; the payoff vector satisfies $w'_j = w_j$ for all $j \in T(y)$, $w'_j = u_j$ for all $j \in A$ and $w'_j = 0$ for all $j \in N\backslash(T(y) \cup A)$. Since (u, x) is conflict-free, there is no pair blocking (w', y') among the agents of A. Because (w, y) is conflict-free, there is no blocking pair formed by two agents in $T(y)$ or an agent in A and an agent in $T(y)$. Finally, if $N\backslash(T(y) \cup A) \neq \varnothing$ then first, no blocking pair exists between an agent in $T(y)$ and an agent in $N\backslash(T(y) \cup A)$ because w' coincides with w for these agents and (w, y) is conflict-free and second, by using the contradiction assumption, $0 + w'_t = u_t \geq a_{qt}$ for all $t \in A$ and for all $q \in N\backslash(T(y) \cup A)$. Therefore, the outcome (w', y') is conflict-free. Since $a_{j^*x(j^*)} > 0$, it follows that either $u_{j^*} > 0$ or $u_{x(j^*)} > 0$. Hence, the outcome (w', y') is a conflict-free extension of (w, y), which is a contradiction because (w, y) is a PO conflict-free outcome.

Once we have shown that there exist some $t^* \in A$ and $q^* \in N\backslash(T(y) \cup A)$ such that $w_{q^*} + u_{t^*} = u_{t^*} < a_{q^*t^*}$, we claim that such q^* necessarily satisfies $q^* \in T(x)$ and $u_{q^*} > 0$. Otherwise, $u_{q^*} = 0$, in which case $u_{q^*} + u_{t^*} < a_{q^*t^*}$ by (5), and then $\{q^*, t^*\}$ would block (u, x), which is not possible because $t^* \in T(x)$ and (u, x) is a conflict-free outcome. Therefore, $q^* \in T(x)\backslash(T(y) \cup A)$. Since $q^* \notin A$, we must have that $p \equiv x(q^*) \in T(x) \cap T(y)$, so

$$u_p = w_p \tag{6}$$

because otherwise we should have that $q^* \in T(x) \cap T(y)$ according to Lemma 2, which would be a contradiction. Furthermore, the fact that $u_{q^*} > 0$ implies that $u_p < a_{pq^*}$, and so $w_p < a_{pq^*}$ by (6), which implies that $\{q^*, p\}$ blocks (w, y) (because $w_{q^*} = 0$), which is a contradiction since $p \in T(y)$. Hence, $x(j^*) \in T(x) \cap T(y)$, and the proof is complete. ■

Lemma 5 shows some property of the set of PO conflict-free outcomes that is also satisfied by the set of corewise-stable outcomes: If (u, x) and (w, y) are PO conflict-free outcomes, then every unmatched player at x has zero payoff at y. Equivalently, if j has a positive payoff under a PO conflict-free outcome (u, x) then j is matched under every PO conflict-free outcome; in particular, j is matched under every corewise-stable outcome.

Lemma 5 *Let (u, x) and (w, y) be PO conflict-free outcomes. Let $j^* \in T(x)\backslash T(y)$. Then $u_{j^*} = w_{j^*} = 0$ and $u_{x(j^*)} = w_{x(j^*)}$.*

Proof. Denote $k^* \equiv x(j^*)$. If $a_{j^*k^*} = 0$, then $u_{j^*} = w_{j^*} = 0$ and $u_{k^*} = 0$. If $k^* \in T(x)\backslash T(y)$ then $w_{k^*} = 0$. Otherwise, we cannot have that $u_{k^*} \neq w_{k^*}$, because Lemma 2 would imply that $j^* \in T(x) \cap T(y)$, which would be a contradiction. Therefore, it is always the case that $u_{x(j^*)} = w_{x(j^*)}$.

Suppose now that $a_{j^*k^*} > 0$. Under Lemma 4, $k^* \in T(x) \cap T(y)$. Then, since $j^* \notin T(y)$ we have that $w_{j^*} = 0$. In addition, we cannot have that $u_{k^*} \neq w_{k^*}$, according to Lemma 2 and the assumption that $j^* \in T(x)\backslash T(y)$, so $u_{k^*} = w_{k^*}$. Now, suppose by way of contradiction that

$u_{j^*} > 0$. Then, $u_{k^*} < a_{j^*k^*}$. Therefore, $w_{j^*} + w_{k^*} = u_{k^*} < a_{j^*k^*}$, so $\{j^*, k^*\}$ blocks (w, y), which is a contradiction because $k^* \in T(y)$ and (w, y) is a conflict-free outcome. Hence, $u_{j^*} = w_{j^*} = 0$ and $u_{x(j^*)} = w_{x(j^*)}$, and the proof is complete. ■

Lemma 5 implies that if (u, x) and (w, y) are PO conflict-free outcomes and $u_j \neq w_j$ (thus, either $j \in T(x)$ or $j \in T(y)$) we must have that $j \in T(x) \cap T(y)$. Also, Lemma 2 implies $x(j) \in T(x) \cap T(y)$ and $y(j) \in T(x) \cap T(y)$. Therefore,

$$\{j \in N; u_j \neq w_j\} \subseteq \{j \in T(x) \cap T(y); x(j) \in T(x) \cap T(y)\} = \{j \in T(x) \cap T(y); y(j) \in T(x) \cap T(y)\}.$$

We can now prove that all PO conflict-free payoffs are equally efficient

Proposition 10 *Let (u, x) and (w, y) be PO conflict-free outcomes. Then, $\sum_N u_j = \sum_N w_j$.*

Proof. Set

$$B_1(x) = \{j \in T(x) \cap T(y); x(j) \in T(x) \cap T(y)\};$$
$$B_1(y) = \{j \in T(x) \cap T(y); y(j) \in T(x) \cap T(y)\};$$
$$B_2(x) = \{j \in T(x) \cap T(y); x(j) \in T(x) \backslash T(y)\};$$
$$B_2(y) = \{j \in T(x) \cap T(y); y(j) \in T(y) \backslash T(x).$$

Clearly,

$$T(x) \cap T(y) = B_1(x) \cup B_2(x) = B_1(y) \cup B_2(y). \tag{7}$$

Under Remark 3,

$$\sum_{B_1(x)} u_j = v(B_1(x)) \text{ and } \sum_{B_1(y)} w_j = v(B_1(y)). \tag{8}$$

On the other hand, according to Lemma 5, $u_j = w_j$ for all $j \in B_2(x)$ and $w_j = u_j$ for all $j \in B_2(y)$, so

$$\sum_{B_2(x)} u_j = \sum_{B_2(x)} w_j \text{ and } \sum_{B_2(y)} w_j = \sum_{B_2(y)} u_j. \tag{9}$$

Moreover, Lemma 5 implies that

$$\sum_{T(x)\backslash T(y)} u_j = 0 \text{ and } \sum_{T(y)\backslash T(x)} w_j = 0. \tag{10}$$

Therefore, we can write,

$$\sum_N u_j = \sum_{T(x)} u_j + \sum_{N\backslash T(x)} u_j = \sum_{T(x)} u_j = \sum_{T(x)\cap T(y)} u_j + \sum_{T(x)\backslash T(y)} u_j = \sum_{T(x)\cap T(y)} u_j =$$
$$\sum_{B_1(x)} u_j + \sum_{B_2(x)} u_j = v(B_1(x)) + \sum_{B_2(x)} w_j \leq \sum_{B_1(x)} w_j + \sum_{B_2(x)} w_j =$$
$$\sum_{T(x)\cap T(y)} w_j = \sum_{T(x)\cap T(y)} w_j + \sum_{T(y)\backslash T(x)} w_j = \sum_{T(y)} w_j = \sum_{T(y)} w_j + \sum_{N\backslash T(y)} w_j = \sum_N w_j,$$

where the fourth equality uses (10); the fifth equality follows from (7); the sixth equality follows from (8) and (9); the inequality follows from the fact that $B_1(x) \subseteq T(y)$ and y is a conflict-free matching, and so $B_1(x)$ cannot block y; the seventh equality follows from (7); and the eighth equality follows from (10).

Then,

$$\sum_N u_j \leq \sum_N w_j. \tag{11}$$

By reverting the roles between (u, x) and (w, y) in the expression (11) we obtain

$$\sum_N w_j \leq \sum_N u_j. \tag{12}$$

According to (11) and (12) we get that $\sum_N u_j = \sum_N w_j$ and we have completed the proof. ■

Proposition 10 implies that every payoff in S^* reaches the maximum total payoff among all conflict-free payoffs. Together with Corollary 3, Proposition 10 also implies that every payoff in S^* dominates every conflict-free payoff not in S^*.

We will refer to a matching that is compatible with a PO conflict-free payoff as **quasi-optimal**. Proposition 11 asserts that every quasi-optimal matching is compatible with any PO conflict-free outcome. That is, Proposition 11 states for the set of PO conflict-free outcomes a property similar to that stated in Proposition 2 for the set of corewise-stable outcomes. Indeed, the set of PO conflict-free outcomes is the Cartesian product of the set of PO conflict-free payoffs and the set of quasi-optimal matchings.

Proposition 11 *Let (u, x) be a PO conflict-free outcome. Then, x is compatible with any PO conflict-free payoff.*

Proof. Let (w, y) be any PO conflict-free outcome. We want to show that $w_i + w_j = a_{ij}$ if $x(i) = j$ and $w_i = 0$ if $i \in U(x)$. Notice that

$$\sum_{T(x)} u_i = \sum_{T(x)} u_i + \sum_{U(x)} u_i = \sum_N u_i = \sum_N w_i = \sum_{T(x)} w_i + \sum_{U(x)} w_i =$$
$$\sum_{T(x)} w_i + \sum_{T(y)\backslash T(x)} w_i + \sum_{U(y)\backslash T(x)} w_i = \sum_{T(x)} w_i,$$

where the third equality is due to Proposition 10; and Lemma 5 was used in the last equality to conclude that $\sum_{T(y)\backslash T(x)} w_i = 0$. Then,

$$\sum_{T(x)} u_i = \sum_{T(x)} w_i. \tag{13}$$

We can write $T(x) = (T(x) \cap T(y)) \cup (T(x)\backslash T(y))$. From Lemma 5 it follows that $\sum_{T(x)\backslash T(y)} u_i = \sum_{T(x)\backslash T(y)} w_i = 0$. Then, $\sum_{T(x)} u_i = \sum_{T(x)\cap T(y)} u_i$ and $\sum_{T(x)} w_i = \sum_{T(x)\cap T(y)} w_i$. Therefore, using (13) we obtain

$$\sum_{T(x)\cap T(y)} u_i = \sum_{T(x)\cap T(y)} w_i. \tag{14}$$

To prove that $w_i + w_j = a_{ij}$ if $x(i) = j$, set $G \equiv \{\{i,j\} \subseteq T(x) \cap T(y); x(i) = j\}$ and $H \equiv \{\{i,j\} \subseteq T(x); i \in T(x) \cap T(y), j \in T(x) \backslash T(y)$ and $x(i) = j\}$. Lemma 5 implies that

$$\sum_H u_i = \sum_H w_i. \tag{15}$$

Moreover, since w is conflict-free, we must have that $w_i + w_j \geq a_{ij}$ for all $\{i,j\} \subseteq T(x) \cap T(y)$. Then, in particular,

$$\sum_G a_{ij} \leq \sum_G (w_i + w_j). \tag{16}$$

Therefore,

$$\sum_{T(x)\cap T(y)} u_i = \sum_G (u_i + u_j) + \sum_H u_i = \sum_G a_{ij} + \sum_H w_i \leq \sum_G (w_i + w_j) + \sum_H w_i = \sum_{T(x)\cap T(y)} w_i,$$

where we used (15) in the second equality and (16) in the inequality. According to (14), the inequality must be an equality, and so we have proved that $w_i + w_j = a_{ij}$ for all $\{i,j\}$ such that $\{i,j\} \subseteq (T(x) \cap T(y))$ and $x(i) = j$. Next, consider $\{i,j\}$ with $x(i) = j$ such that either $i \in (T(x)\backslash T(y))$ or $j \in (T(x)\backslash T(y))$. Without loss of generality suppose that $i \notin T(y)$. Then, $w_i = 0$ and, under Lemma 5, we have that $u_i = 0$ and $w_j = u_j$, from which follows that $w_i + w_j = u_i + u_j = a_{ij}$, so $w_i + w_j = a_{ij}$.

It remains to show that $w_i = 0$ for all $i \in U(x)$. But this is immediate from the fact that if $i \in T(y)\backslash T(x)$, then Lemma 5 implies that $w_i = 0$. Hence, the matching x is compatible with w and we have completed the proof. ■

Our final result in this section provides another feature that is shared by all PO conflict-free outcomes. It also helps us better understand the structure of the PO conflict-free and that of the conflict-free outcomes. It states that if a PO conflict-free outcome is not corewise-stable then the set of pairs of blocking agents is the same in every PO conflict-free outcome. Moreover, each of those pairs also blocks any conflict-free outcome. This result implies, in particular, that if an agent is unmatched at some PO conflict-free outcome but he is matched with zero payoff at another PO conflict-free outcome then that agent will never be part of a blocking pair in a PO conflict-free outcome. And since every conflict-free outcome is extended by a PO conflict-free outcome, any blocking agent of a PO conflict-free outcome is unmatched at any conflict-free outcome.

Proposition 12 *Let $(u,x) \in S^*\backslash C$ and let $\{j,k\}$ be a blocking pair for (u,x). Then, $\{j,k\}$ blocks (w,y), for any $(w,y) \in S$. In particular, $\{j,k\} \subseteq U(y)$, for any $(w,y) \in S$.*

Proof. Notice first that, given that $\{j,k\}$ blocks (u,x), it is the case that j and k are unassigned at x, so $0 = u_j + u_k < a_{jk}$. Moreover, j and k have a zero payoff at any PO conflict-free outcome, under Lemma 5 (even if j or k were matched in a PO conflict-free outcome, they would obtain a zero payoff). Therefore, the sum of the payoffs of j and k in any PO conflict-free outcome is less than a_{jk}, so $\{j,k\}$ blocks any PO conflict-free outcome. Now, use propositions 5 and 9 to get that any (w, y) is extended by some PO conflict-free outcome, so $\{j, k\}$ blocks any conflict-free (w, y). In particular, j and k are unassigned at y. Hence, the proof is complete. ∎

The properties that we have proved in this section suggest that the set of PO conflict-free outcomes constitutes a natural solution concept if one cares about payoffs that are "as stable as possible." First, every PO conflict-free outcome is internally stable, so no active player has an incentive to look for other partners inside or outside the set of active players. Second, each PO conflict-free outcome provides the maximum surplus out of the set of internally stable outcomes; and all the internally stable outcomes outside the set provide less surplus. Third, any internally stable outcome that is not PO conflict-free can be naturally extended to a PO conflict-free outcome. Fourth, all PO conflict-free outcomes are compatible with the same matchings. Fifth, all the previous properties of PO conflict-free outcomes replicate properties that are satisfied by the corewise-stable outcomes. Finally, the set of PO conflict-free outcomes is always non-empty and, as will be proved in the next section, the set of PO conflict-free payoffs coincides with the core, when the core is non-empty.

The intuitive idea of a PO conflict-free outcome is that it corresponds to an outcome that we can expect to occur in an idealized environment where agents take decisions under the assumption of cooperative behavior. In fact, the properties of the conflict-free outcomes allow us to envision a dynamic and finite partnership formation process of conflict-free outcomes that ends with a PO conflict-free outcome. At every step t of this process, the current unmatched agents work among themselves to form partnerships and to split the gains obtained in these partnerships. Once the vector of payoffs of any new partnership of matched agents is established, the unmatched agents are not interested in trading with any matched agent, currently and previously formed. Because of the properties concerning the extensions of conflict-free outcomes (Proposition 9 and Corollary 4) and the fact that the PO conflict-free payoffs extend the conflict-free payoffs (Proposition 5), this process always ends in a PO conflict-free outcome, which provides further support for the set of PO conflict-free outcomes as a natural solution concept for the one-sided assignment game.

To describe the sequential process, consider any conflict-free outcome (u, x) such that $T(x) \neq \varnothing$. Define

$$A(u, x) \equiv \{(u^p, x^p) \text{ is conflict-free}; T(x^p) \subseteq T(x), \text{ and } x^p(j) = x(j) \text{ and } u^p_j = u_j \text{ for all } j \in T(x^p)\}.$$

That is, $A(u, x)$ is the set of conflict-free outcomes in which the pairs are matched according to x and have the same payoffs as u.

Denote $B_1(u, x) \equiv \{S^p \subseteq T(x); S^p \neq \varnothing \text{ and } S^p = T(x^p) \text{ for some } (u^p, x^p) \in A(u, x)\}$. That is, $B_1(u, x)$ is the set of the non-empty stable active coalitions of the conflict-free outcomes in $A(u, x)$. The set $B_1(u, x)$ is non-empty, since $T(x) \in B_1(u, x)$. Furthermore, $B_1(u, x)$ is finite and is endowed with the partial order defined by the set inclusion relation. Then $B_1(u, x)$ has a minimal element, that is, there exists some coalition that does not have any sub-coalition in $B_1(u, x)$. Set

$C_1(u,x)$ any such coalition[9] and let (u^1,x^1) be the corresponding conflict-free outcome in $A(u,x)$.

If $C_1(u,x) \neq T(x)$, that is, if $(u^1,x^1) \neq (u,x)$, we denote $B_2(u,x) \equiv \{S^p \subseteq T(x); S^p = T(x^p)$ for some $(u^p,x^p) \in A(u,x)$ such that (u^p,x^p) is an extension of $(u^1,x^1)\}$. The set $B_2(u,x)$ is also non-empty because $T(x) \in B_2(u,x)$. By using similar arguments as above, we obtain the existence of a minimal element of $B_2(u,x)$. Set $C_2(u,x)$ any such coalition and let (u^2,x^2) be the corresponding conflict-free outcome in $A(u,x)$. By construction, (u^2,x^2) is a conflict-free outcome in $A(u,x)$ and it extends (u^1,x^1).

By continuing this procedure, we obtain a finite sequence of conflict-free outcomes in $A(u,x)$: $(u^1,x^1),(u^2,x^2),\ldots,(u^k,x^k)$, where $(u^k,x^k)=(u,x)$ and (u^{p+1},x^{p+1}) extends (u^p,x^p) for all $p = 1,\ldots,k-1$. Then, $T(x) = C_k(u,x)$.

We can describe the sequential process generated by $(C_p(u,x))_{p=1,\ldots,k}$ as follows. The first step of such a process yields (u^1,x^1), the second step yields (u^2,x^2), and so on. At every step t, no agent j in $N\backslash C_t(u,x)$ is willing to pay any agent i in $C_t(u,x)$ more than u_i^t. Thus, at any step t, the current outcome (u^t,x^t) is conflict-free and is an extension of the current outcome (u^{t-1},x^{t-1}).

If the conflict-free outcome (u,x) is not a PO conflict-free outcome then there is another conflict-free outcome that extends (u,x) and the procedure could continue. It only ends when no interaction is able to benefit the agents involved, in which case the core is reached, or when any new interaction leads to a set of matched agents which is not internally stable. In any case, the final outcome is a PO conflict-free outcome. Since any PO conflict-free outcome can be formed this way and those outcomes cannot be extended by another conflict-free outcome, we have that the set of PO conflict-free outcomes is the set of all the outcomes which constitute the final steps of such procedures.

The following example shows that the number of processes that reach a PO conflict-free outcome may vary inside the set of PO conflict-free outcomes. This happens even though, in the example there is only one matching which is compatible with all the PO conflict-free payoffs.

Example 2 *The set of players is $N = \{1,2,3,4\}$ and the surplus of the partnerships is $a_{12} = 10$, $a_{13} = 4$, $a_{34} = 12$, and $a_{ij} = 0$ for the other partnerships. The set of PO conflict-free outcomes, which coincides with the corewise-stable outcomes, is the set of outcomes (u,x) that satisfy $x_{12} = 1$, $x_{34} = 1$ and the payoffs are non-negative numbers with $u_1+u_2 = 10$, $u_3+u_4 = 12$, and $u_1+u_3 \geq 4$.*

(i) For the PO conflict-free outcome with $u = (6,4,5,7)$, there are two different processes: either ($u^1 = (6,4,0,0)$, x^1 with $x^1_{12} = 1$ and the other entries are 0) or ($u^1 = (0,0,5,7)$, x^1 with $x^1_{34} = 1$ and the other entries are 0); in both cases $(u^2,x^2) = (u,x)$.

(ii) For the PO conflict-free outcome with $u = (6,4,3,9)$, there is only one process: ($u^1 = (6,4,0,0)$, x^1 with $x^1_{12} = 1$ and the other entries are 0) and $(u^2,x^2) = (u,x)$. Note that ($u^1 = (0,0,3,9)$, x^1 with $x^1_{34} = 1$ and the other entries are 0) cannot be part of the process because it is not a conflict-free outcome: players 1 and 3 block this outcome.

(iii) Finally, for the PO conflict-free outcome with $u = (3,7,3,9)$, the only process is the trivial one-step process: $(u^1,x^1) = (u,x)$. ■

[9] There can be several minimal coalitions.

We will come back to the partnership formation process in the next section, once we analyze the relationship between the set of PO conflict-free outcomes and the core.

5 Pareto-optimal conflict-free outcomes and corewise-stable outcomes

The previous section states several appealing properties of the set of PO conflict-free outcomes. They allowed us to propose this set as a natural solution concept for the one-sided assignment game. In the current section, we further support our proposal as a new stability concept by showing that the set of PO conflict-free payoffs and the core coincide, when the core is not empty. Moreover, the relationship between the core and the set of PO conflict-free outcomes constitutes a useful tool to establish conditions under which the core is non-empty in these environments.

Before proving the equivalence, we will state other properties that show some relationships between corewise-stable outcomes and conflict-free outcomes.

Lemma 6 *Suppose the set of corewise-stable outcomes is not empty. Let x^* be an optimal matching. Then $S(x^*) = C$.*

Proof. Under Proposition 2, every corewise-stable payoff is compatible with x^*. Since $C \neq \varnothing$ and $C \subseteq S$, we have that x^* is a conflict-free matching, so $S(x^*)$ is well defined. Let (w, x^*) be some corewise-stable outcome. We will prove that $S(x^*) \subseteq C$ by way of contradiction. Then, take $u \in S(x^*)$ and suppose that $\{i, j\}$ blocks u. Since u is conflict-free, we have that i and j are unmatched at x^*, so $u_i = u_j = w_i = w_j = 0$. Then, $w_i + w_j = u_i + u_j < a_{ij}$, which contradicts the corewise-stability of w. The other direction is immediate from the fact that every corewise-stable payoff is compatible with x^*, under Proposition 2, and it is conflict-free. ■

Proposition 13 proves two important properties of the conflict-free outcomes when the core is non-empty. First, it states that for any conflict-free outcome which is not corewise-stable, it is always possible to construct a new outcome that keeps the payoff of each matched player and is corewise-stable. Second, it shows that the sum of the payoffs of the set of agents that are matched in a conflict-free outcome is always maintained in any corewise-stable outcome.

Proposition 13 *Let (u, x) be a conflict-free outcome which is not corewise-stable. Suppose the set of corewise-stable outcomes is non-empty. Then:*
(a) there exists a corewise-stable outcome (u^*, z) that extends (u, x), and
(b) $\sum_{T(x)} u_i = \sum_{T(x)} w_i$ for all $w \in C$.

Proof. According to Proposition 4, the set of active partnerships of x is part of some optimal matching. Therefore, there is some optimal matching z such that $z(i) = x(i)$ for all $i \in T(x)$. Let (w, z) be any corewise-stable outcome. Construct the outcome (u^*, z) such that $u_i^* = u_i$ for all $i \in T(x)$ and $u_i^* = w_i$ for all $i \in N \backslash T(x)$. The outcome (u^*, z) is feasible. We claim that $u^* \in C$.

In fact, suppose $\{i,j\}$ blocks u^*. Then, $u_i^* + u_j^* < a_{ij}$. Notice that, by construction, $u^* \geq u$, so $u_i + u_j < a_{ij}$. Since x is conflict-free, we must have that $\{i,j\} \subseteq N \backslash T(x)$. On the other hand, the corewise-stability of w implies that $u_i^* + u_j^* = w_i + w_j \geq a_{ij}$, which contradicts the assumption that $\{i,j\}$ blocks u^*. Then, u^* does not have any blocking pair.

The property that (u^*, z) is individually rational is immediate from the individual rationality of (u,x) and (w,z). According to Definition 6, it remains to show that $v(N) = \sum_N u_i^*$. Write:

$$v(N) \leq \sum_N u_i^* = \sum_{T(x)} u_i + \sum_{N \backslash T(x)} w_i = v(T(x)) + \sum_{N \backslash T(x)} w_i \leq \sum_{T(x)} w_i + \sum_{N \backslash T(x)} w_i = \sum_N w_i = v(N), \tag{17}$$

where in the first inequality we used the fact that u^* does not have any blocking pair, in the second equality we used Remark 3, and the second inequality follows from the corewise-stability of w. Then, the inequalities in (17) must be equalities, so $v(N) = \sum_N u_i^*$. Therefore, we have proved that (u^*, z) is corewise-stable.

To see that (u^*, z) extends (u,x), use that $u_i^* \geq u_i$ for all $i \in N$. Given that (u^*, z) is corewise-stable and that (u,x) is unstable, we have that $\{j \in N; u_j^* > u_j\} \neq \varnothing$. On the other hand, since $u_j^* = u_j$ for all $j \in T(x)$, it follows that $\{j \in N; u_j^* > u_j\} \subseteq N \backslash T(x)$. Then, according to Definition 9, (u^*, z) extends (u,x), and we have proved part (a) of the proposition.

Now use that the inequalities in (17) must be equalities, so $\sum_{T(x)} u_i = v(T(x)) = \sum_{T(x)} w_i$, which proves assertion (b). Hence, we have completed the proof. ■

It is worth mentioning that while Proposition 13 states that any conflict-free outcome which is not in the core can be extended to a corewise-stable payoff, it does not ensure that it is possible to "shrink" any corewise-stable payoff. For example, in Case (iii) of Example 2, the payoff vector $u = (3,7,3,9)$ is a corewise-stable payoff but $(3,7,0,0)$ is not a conflict-free payoff, even though $x_{12} = 1$ in the corewise-stable matching.

The set of PO conflict-free payoffs provides a set of solutions for every game. Theorem 1 states that this set coincides with the core, which is equivalent to the set of pairwise-stable payoffs, if and only if the core is not empty.

Theorem 1 *The set of corewise-stable payoffs is non-empty if and only if $S^* = C$.*

Proof. Suppose the core is non-empty. Let (u,x) be a Pareto-optimal conflict-free outcome. We are going to show that (u,x) is corewise-stable. In fact, suppose by way of contradiction, that (u,x) is unstable. Under Proposition 13, there is some corewise-stable outcome (u^*, z) which extends (u,x). Then, $u_j^* \geq u_j$ for all $j \in N$ and $u_j^* > u_j$ for at least one player j. But this contradicts the fact that (u,x) is a PO outcome. Hence, (u,x) is corewise-stable. In the other direction, let (u,x) be a corewise-stable outcome. Then, (u,x) is PO feasible. Since every conflict-free outcome is feasible, it follows that there is no conflict-free Pareto improvement of (u,x). Given that (u,x) is conflict-free, it must be a PO conflict-free outcome. Hence, $S^* = C$.

The proof that $S^* = C$ implies that the core is not-empty is immediate from the fact that $S^* \neq \varnothing$. ■

To emphasize that the set of PO conflict-free outcomes is the natural extension of the set of corewise-stable outcomes when the last set is empty, let us mention that we can use properties for the set of PO conflict-free outcomes, together with Theorem 1, to obtain the corresponding properties of the core as immediate corollaries. In particular, the result that an optimal matching is compatible with any corewise-stable payoff (Proposition 2) is a corollary of Proposition 11 and Theorem 1 for the environments where the core is not empty, once we realize that the quasi-optimal matchings are optimal if the core exists. And Corollary 1 is just a corollary of that result.

Going back to the partnership formation process discussed in the previous version, Theorem 1 ensures that it ends when a core outcome is reached, whenever the core is non-empty. If the final outcome of some sequential process is corewise-stable then the final outcome of every sequential process is corewise-stable. On the other hand, whatever sequence is formed, if the final outcome of some coalition formation process is corewise-unstable then the final outcome of every coalition formation process is corewise-unstable and the core is empty.

We can use the relationship between the core and the set of PO conflict-free outcomes established in Theorem 1 to obtain conditions under which the core exists. First, Theorem 2 uses the properties of the set of PO conflict-free payoffs to provide a necessary and sufficient condition for the core to be non-empty based on the examination of PO conflict-free payoffs.

Theorem 2 *The set of corewise-stable outcomes is non-empty if and only if every PO conflict-free payoff is PO feasible.*

Proof. Suppose first that the set of corewise-stable outcomes is non-empty and let $u \in S^*$. Theorem 1 implies that $u \in C$, so u is PO feasible.

In the other direction, take a PO conflict-free payoff u, which is also PO feasible, and let x be a conflict-free matching compatible with u. Then, $\sum_N u_i = \sum_x (u_i + u_j) = \sum_x a_{ij}$. Since u is PO feasible then $v(N) = \sum_N u_i$. Therefore, $\sum_x a_{ij} = v(N)$, so x is an optimal matching. Proposition 5 then implies that $u \in C$, so $C \neq \varnothing$. Hence, the proof is complete. ∎

Our final theorem provides a necessary and sufficient condition for the emptiness of the core based on the idea of "non-solvable blocking pairs."

Some of the blocking pairs of a conflict-free outcome "vanish" along the partnership formation process that we have described at the end of the previous section, in the sense that they do not block some conflict-free outcomes that extend the original conflict-free outcome. Other blocking pairs "persist" along the process as they block all the conflict-free extensions of the original outcome, including the PO conflict-free outcomes that can be obtained in the last term of the sequences. As we will show in Theorem 3, the last type of blocking pairs play a fundamental role in the emptiness of the core. We will call them "non-solvable blocking pairs."

Definition 12 *Let (u, x) be a conflict-free outcome and let $\{i, j\} \subseteq U(x)$, with $a_{ij} > 0$ (i.e., $\{i, j\}$ is a blocking pair). We say that* ***$\{i, j\}$ is a non-solvable blocking pair of (u, x)*** *if either $u \in S^*$ or $\{i, j\} \subseteq U(x')$ for every conflict-free extension (u', x') of (u, x). Also, we say that* ***$\{i, j\}$ is a***

non-solvable blocking pair *if it is a non-solvable blocking pair for some conflict-free outcome* (u, x).

Therefore, since $a_{ij} > 0$, **if $\{i, j\}$ is a non-solvable blocking pair for (u, x)**, then $\{i, j\}$ blocks every conflict-free extension of (u, x), if any. In this case, **$\{i, j\}$ also blocks the PO conflict-free outcome which extends (u, x)**, and that PO conflict-free outcome is corewise-unstable which, under Theorem 1, implies $C = \varnothing$. In fact, **every blocking pair $\{i, j\}$ of a PO conflict-free outcome is a non-solvable blocking pair of all conflict-free outcomes**. This is because, under Proposition 12, the pair $\{i, j\}$ blocks every conflict-free outcome (including all PO conflict-free outcomes). Thus, $\{i, j\}$ is a non-solvable blocking pair of every conflict-free outcome. Hence, **the set of non-solvable blocking pairs of a given conflict-free outcome is the same as that of every conflict-free outcome and**, in particular, **it coincides with the set of blocking pairs of any PO conflict-free outcome**. These conclusions are formalized in the following results.

Proposition 14 *Let* (u, x) *be a conflict-free outcome and let* $\{i, j\}$ *be a non-solvable blocking pair for* (u, x)*. Then,* $\{i, j\}$ *is a non-solvable blocking pair of every conflict-free outcome.*

Proof. As stated in Definition 12, $\{i, j\}$ is a blocking pair of the PO conflict-free outcome that extends (u, x), in case $u \notin S^*$. Then, in any case, $\{i, j\}$ is a blocking pair of a PO conflict-free outcome. According to Proposition 12, $\{i, j\}$ is a blocking pair of every conflict-free outcome, and is so for every extension of any conflict-free outcome. Definition 12 then implies that $\{i, j\}$ is a non-solvable blocking pair of every conflict-free outcome. Hence, the proof is complete. ■

Corollary 5 uses Proposition 14 to characterize the non-solvable blocking pairs as the blocking pairs of a PO conflict-free outcome.

Corollary 5 *The pair* $\{i, j\}$ *is a non-solvable blocking pair if and only if it is a blocking pair of a PO conflict-free outcome.*

Proof. According to Definition 12, the non-solvable blocking pairs of a PO conflict-free outcome are its blocking pairs. On the other hand, under Proposition 14, the set of non-solvable blocking pairs of a given conflict-free outcome is the same as that for every conflict-free outcome, in particular for every PO conflict-free outcome. Then, the set of non-solvable blocking pairs of a given conflict-free outcome coincides with the set of blocking pairs of any PO conflict-free outcome. ■

We can now state the theorem that provides the conditions for the existence of the core.

Theorem 3 *The following conditions are equivalent:*
(i) $C = \varnothing$*;*
(ii) every conflict-free outcome has a non-solvable blocking pair;
(iii) there is a conflict-free outcome that has a non-solvable blocking pair.

Proof. Suppose $C = \varnothing$. Let (w, y) be a PO conflict-free outcome. Since $C = \varnothing$ we have that (w, y) is corewise-unstable according to Theorem 1. Let $\{i, j\} \subseteq U(y)$ with $a_{ij} > 0$. It follows from Proposition 12 that $\{i, j\}$ blocks every conflict-free outcome, in particular it blocks every extension of any conflict-free outcome, if any. Then, under Definition 12, $\{i, j\}$ is a non-solvable blocking pair of every conflict-free outcome. Then (i) implies (ii). Clearly, (ii) implies (iii).

Now, let (u, x) be a conflict-free outcome and suppose $\{i, j\}$ is a non-solvable blocking pair for (u, x). As shown in the proof of Proposition 14, $\{i, j\}$ is a blocking pair of a PO conflict-free outcome. Theorem 1 implies that $C = \varnothing$, so (iii) implies (i). Hence, we have completed the proof. ■

Remark 7 *From the results above we can conclude that $\{i, j\}$ is a non-solvable blocking pair for some conflict-free outcome if and only if the pair $\{i, j\}$ blocks every conflict-free outcome. Then, if two conflict-free outcomes have disjoint sets of blocking pairs, the core is non-empty.* ■

Our final remark makes clear the extent to which a non-solvable blocking pair is distinct from the other blocking pairs.

Remark 8 *Our previous results imply that if $\{i, j\}$ is a non-solvable blocking pair for (u, x) then there is no conflict-free extension (w, y) of (u, x) such that $y(i) = j$.* ■

6 Concluding remarks

Our paper studies the one-sided assignment game, which is the generalization of the two-sided assignment game of Shapley and Shubik (1972) to the case where any two agents can form a partnership. It provides a new point of view about stability through the concepts of conflict-free outcome and the Pareto-optimal conflict-free outcome.

Conflict-free outcomes capture some notion of internal stability: In a conflict-free outcome, a matched agent cannot block the situation deviating with either another matched or unmatched agent. In that sense, the set of matched agents (the members of the "club of active agents") is in a stable situation as none of its members can deviate. The properties of the set of conflict-free outcomes allow us to propose a dynamic "club formation" process. The club enlarges at each step of the process, but the payoff of the old members does not change with the arrival of new members. At the end of the process, we always obtain an outcome which is Pareto-optimal in the set of conflict-free outcomes.

We view the set of Pareto-optimal conflict-free outcomes as a natural solution concept for the one-sided assignment game. Each of them generates (and they are the only ones that do so) the highest possible total surplus in the set of conflict-free outcomes. And, as the previous dynamic process suggests, these outcomes are "as stable as possible," in the sense that any matching involving a larger set of matched agents will necessarily be unstable; the club of active agents would be too large. In fact, the set of Pareto-optimal conflict-free payoffs coincides with the core when the core is not empty. Thus, the solution concept keeps all the good properties of the core when it exists, but it also provides a prediction for those markets where the core does not exist. Moreover, several of the nice properties of the core, when it is non-empty, are extended to the set of Pareto-optimal conflict-free outcomes.

Bondareva (1963) and Shapley (1967) proved that the core of a transferable utility game is non-empty if and only if the game is balanced. Thus, for the game considered here, the condition that every PO conflict-free payoff is PO feasible is equivalent to balancedness. This suggests the question of whether this equivalence persists in all transferable-utility (TU) games. The answer to this question is not easy. Our results strongly rely on the existence of a feasible matching underlying every feasible outcome. However, players do not necessarily form partnerships in the general TU game. On the other hand, the intuition behind a conflict-free outcome is not related to a matching and seems to be quite general: if all "interactions" are made under the premise of optimal behavior, a conflict-free outcome results. This suggests that, by conveniently adapting the concept of a conflict-free outcome, the desired equivalence can be obtained for a general TU game.

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