Daunou's Voting Method

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Abstract

Pierre Daunou, a contemporary of Borda and Condorcet during the era of the French Revolution and active debates on alternative voting rules, proposed a method that chooses the strong Condorcet winner if there is one, otherwise eliminates Condorcet losers and uses plurality voting on the remaining alternatives. We axiomatically characterize his method which combines potentially conflicting criteria of majoritarianism by ordering them lexicographically. This contribution serves not just to remind ourselves that a 19^{th}-century vintage may still retain excellent aroma and taste, but also to open up a novel way of applying potentially conflicting desiderata by accommodating them lexicographically. *Journal of Economic Literature* Classification Nos.: D71, D72.

A study of the history of opinion is a necessary preliminary to the emancipation of the mind. I do not know which makes a man more conservative—to know nothing but the present, or nothing but the past.

John Maynard Keynes, *The End of Laissez-Faire*, 1926.

1 Introduction

This paper provides an axiomatic characterization of a voting rule proposed by Pierre Daunou in his 1803 Paper on Elections by Ballot. An interesting feature of his method is that it combines two different (possibly conflicting) desiderata based on Marquis de Condorcet's (1785) proposals and the plurality rule. Daunou gives priority to the Condorcet winner criterion and the Condorcet loser criterion and then uses the plurality rule in situations where these two principles do not apply. Our characterization of Daunou's voting method brings the nature of his proposal into clearer relief, arguing that none of the excellent aroma and flavor of his 19^{th}-century vintage has been lost over the last two centuries. In addition (and perhaps more importantly), the novel technique presented here promises to render some service that goes well beyond an appreciation of a historically highly significant proposal. By combining potentially incompatible criteria lexicographically, we succeed in avoiding impossibilities and come up with a reasonable compromise. This variant of the axiomatic method has the potential of playing a useful and fundamental role in many different contexts within economic theory.

Daunou's contribution dates from a period of deep changes in Europe that were accompanied by active debates regarding all aspects of political philosophy and, in particular, the design of new voting rules. The intellectual atmosphere during the last part of the 18^{th} and the early 19^{th} centuries not only provided the theoretical foundations for political change, but also received feedback from it, in the form of opportunities to experiment with new methods to express the collective views of society. Great and well-known contributors to such debates were Rousseau and Montesquieu, on the philosophical side, while Jean-Charles de Borda (1781) and Condorcet (1785), two achieved thinkers and mathematicians, were important proposers of voting systems that, in spite of some precedents, were perceived not only as being new, but also opposed to each other.

Condorcet proposed the choice of the candidate who would defeat all others in pairwise voting whenever there is such a candidate (this requirement is known as the Condorcet winner criterion) and completed the description of his rule by a hard-to-interpret proposal for the case where such a winner does not exist. In contrast, Borda suggested to choose those candidates who would accumulate more individual wins over other candidates, added across all voters; this criterion can alternatively be expressed as the maximization of the points assigned to each alternative on the basis of its position in the rankings of the different voters.

The two proposals fundamentally diverge. The rule advocated by Daunou, that we shall examine here, partially deviates from each of these two methods. Yet, all three start from a shared agreement about the shortcomings of the plurality rule—the prevailing voting rule at the time. One of their most profound criticisms regarding plurality voting was that it did not always appropriately respond to the majoritarian principle when more than two alternatives were at stake and no alternative obtained an absolute plurality. Indeed, selecting an alternative that merely has a relative plurality may lead to numerous anomalies. Avoiding two of these anomalous consequences of the plurality rule is the clear objective of Daunou's proposal. Yet, his approach was less radical than Borda's, who proposed a completely new method. Daunou's corrects the major shortcomings of plurality but then reverts again to this traditional procedure in those cases where its flaws are taken care of. Because of that, our axiomatization suggests a new way to approach the study of different voting rules in modern terms. Unlike most characterizations that require each axiom to hold on the whole domain of the rule, we follow Daunou's suit and propose to classify the axiomatic requirements lexicographically, in terms of importance, and then allow for other axioms to be added to the fundamental ones in those cases where the major requirements are no longer operational. This novel procedure allows us to characterize Daunou's rule, which essentially consists of respecting the Condorcet principles when they apply, and then using plurality only in those cases where the higher-level principles are no longer effective. We think that this new approach holds promise for further applications and opens the way to a non-maximizing approach to mechanism design.

Condorcet, Borda and Daunou were members of the French National Institute of Sciences and Arts, an institution that became a living laboratory where to test, in the small, the virtues and the vices of different voting methods. Actually, the Borda method was adopted for the election of new members in 1796 in response to Daunou's proposal to that effect; the Institute terminated the use of the Borda rule in 1804 as a response to intense pressure that it was subjected to by Napoleon Bonaparte. However, Daunou later became a strong critic of the method, as is apparent in his 1803 text. There he attacked that method on different grounds, both directly and by criticizing the arguments put forward in its defense by José-Isidoro Morales, the author of a Mathematical Memoir on the Calculation of Opinion in Elections (1797) that defended the use of Borda's rule and was well appreciated by different members of the Institute, including Borda.

Daunou introduces his proposal both in negative and positive terms. He first postulates a number of maxims about the problems that should be avoided when proposing a voting method, and states that (Daunou, 1803; 1995, p. 251) "...we shall declare faulty all election methods

> [(i)]—which have a tendency to correct the election results rather than just counting them,
>
> [(ii)]—or which take into account the supposed intensity of the votes, instead of taking them all to be equal and simply counting them,
>
> [(iii)]—or which allow the election of a candidate decisively rejected by an absolute majority of the voters,
>
> [(iv)]—or which make no distinction between cases in which there is an absolute majority and cases in which there is not,
>
> [(v)]—or finally which allow or facilitate the victory of a minority candidate over one preferred by an absolute majority to all others, taken together or individually."

Later on (Daunou, 1803; 1995, p. 274), he makes a positive proposal for his election method (the voting rule that we shall characterize after some comments and clarifications) in the following terms:

> "Each voter has only to draw up one ballot paper according to Borda's method. Everything else is done by the scrutineers.
>
> They first of all check whether any candidate has obtained an absolute majority of first place votes. If so, they declare him elected.
>
> If not, they check how many times each candidate is preferred to each of his opponents, and if an absolute majority prefers one candidate to each of the others taken individually, that candidate is elected.
>
> If there is still no result, they examine whether any candidate is ranked below all of the others either collectively or individually, on an absolute majority of ballot papers. If so, they eliminate him.
>
> Then, they work out which of the candidates who have not been eliminated obtained the relative plurality of first votes, and this candidate is elected if it is absolutely necessary to elect someone."

Requirements (i) and (ii) in the maxims are the most obscure for the modern reader, and we take them to reflect several of Daunou's concerns. One of these concerns is to guarantee the simplicity of the method. In that sense, the first sentences of his positive proposal, where he describes the ballot as the one according to Borda's method, can just be translated in modern terms as requiring the voters to express their ranking of the alternatives. In the same spirit, also reflected in his proposal, we shall translate these concerns by concentrating on anonymous social choice correspondences. In certain parts of Daunou's work he criticizes Borda's rule, *inter alia,* for providing voters with ample possibilities to distort the results of the election by manipulating the voting scheme via the strategic use of their votes. In fact, this strategic manipulation was one of the reasons why Daunou was dismayed with the performance of the rule that he had recommended to the Institute in the first place. Thus, one of his reasons for expressing (ii) might also be his association between the possibility of manipulation and the use of intensities.

Maxim (iii) is intended to rule out the election of a strong Condorcet loser. Therefore, the Condorcet loser criterion is the natural property to address this. Likewise, maxims (iv) and (v) ensure that if a strong Condorcet winner exists, then this candidate—and only this candidate—should be elected. Thus, the Condorcet winner criterion takes care of postulates (iv) and (v).

It appears that Daunou's (1803) maxims are fully described by the conjunction of anonymity, the Condorcet loser criterion and the Condorcet winner criterion. Daunou (1803; 1995, p. 273) makes a distinction between elections with a large number of individuals and alternatives and those for which these numbers are relatively small. The rule that he proposed in the preceding text and that we shall characterize is the one corresponding to the small-numbers case. He is much less explicit about his proposal for the case of large numbers. In an admittedly arbitrary statement, Daunou distinguishes between those elections in which there are both more than seven candidates and more than fifty voters, and those in which there are either not more than fifty voters or not more than seven candidates.

We take it as a way to illustrate by means of example the increasing complexity of problems related to the filling of ballots and their scrutiny. Indeed, the latter may not be a problem nowadays, but it remains true that asking voters to rank an excessive number of candidates is still very demanding.

The last step in Daunou's proposal reverts to the use of plurality, when all other criteria do not reach a definite choice. He explicitly justifies and explains his proposal on pages 268–269 of Daunou (1803; 1995) as follows:

> "... [when] there is no general will and no majority preference for one of the candidates over the others ... [and] it is absolutely necessary to elect someone ... I consider it best to elect the candidate with the most first place votes: for as soon as we have ascertained that there is no absolute majority, whether clear or hidden, in favor of any of the other candidates, it seems perfectly just to give the authority of this majority to the simplest, most direct, and clearest relative plurality."

Daunou's ambiguous use of the singular form in this quotation needs to be clarified—clearly, there may be multiple plurality winners. We interpret his statement to mean that all of them are to be elected.

In general, we cannot but claim some interpretative license in passing from this descriptive section to the formal analysis that follows. Similar results could be obtained if these interpretations were modified, and we shall comment on these observations as we go along. But, in essence, this is the spirit that is distilled from our reading of Daunou's work: being aware of two major shortcomings of the plurality rule, he proposes the predominance of the Condorcet criterion, as a principle to select a winner and also to discard some definite losers, and then reverts to plurality as a lesser evil when these major drawbacks are not a threat.

To begin our formal analysis, we define voting rules (Section 2) and the properties that we employ (Section 3). We then proceed to an axiomatic characterization of Daunou's rule (Section 4). The axiomatization clearly displays a combination of the plurality rule, in cases where two basic criticisms of it do not apply, along with solutions to those criticisms, when Daunou deems them relevant, through the use of Condorcet's ideas. To accommodate Daunou's modifications, some of the axioms in our characterization are stated conditional on the absence of strong Condorcet winners and strong Condorcet losers. If these axioms are amended by making them apply unconditionally, an alternative axiomatization of the plurality rule emerges. We note that characterizing Daunou's rule represents a challenge because the classical axiomatizations of the plurality rule such as that provided by Stephen Ching (1996) cannot be taken as a starting point for our approach; this is because Ching's proof method involves profiles for which a strong Condorcet loser may be chosen, a feature that is explicitly ruled out according to Daunou's rule. The Appendix contains a discussion of the insufficiency of a weak axiom related to the exclusion of strong Condorcet losers (Part A), a collection of examples that prove the independence of the axioms we use in our main result (Part B), and an illustration of the proof of our theorem (Part C).

2 Voting rules

For any two sets A and B, we use the notation $A \subseteq B$ to indicate that A is a subset of B, and $A \subsetneq B$ means that A is a *strict* subset of B. There is a non-empty and finite set of candidates X and we use $\mathcal{X}$ to denote the set of all non-empty subsets of X. The set of potential voters is the set $\mathbb{N}$ of positive integers. The set of possible societies under consideration is the set $\mathcal{N}$ of all non-empty and finite subsets of $\mathbb{N}$.

An *ordering* is a reflexive, complete and transitive relation on the set X. For all $i \in \mathbb{N}$, $\mathcal{R}_i$ is the set of all antisymmetric orderings on X with typical element R_i. For $i \in \mathbb{N}$, $R_i \in \mathcal{R}_i$, $S \in \mathcal{X}$ and $x \in S$, x is i's top candidate for R_i in S if xR_iy for all $y \in S$. This top candidate is denoted by $t(R_i, S)$. For all $N \in \mathcal{N}$, $\mathcal{R}_N = \prod_{i \in N} \mathcal{R}_i$ is the set of all profiles of antisymmetric orderings for the population N with typical element $R_N = (R_i)_{i \in N}$. Furthermore, we define $\mathcal{R} = \cup_{N \in \mathcal{N}} \mathcal{R}_N$.

Let $S \in \mathcal{X}$ and $N \in \mathcal{N}$ be such that $|N| = |S|$; that is, the number of voters is equal to the number of candidates. A profile $R_N \in \mathcal{R}_N$ is a *completely symmetric profile* if each candidate is placed in each possible position exactly once in the profile R_N; see also Ching (1996, p. 300). If $|N| = m|S|$ for some $m \in \mathbb{N}$ (that is, if the number of voters is a multiple of the number of candidates), a profile $R_N \in \mathcal{R}_N$ is an *m-fold replica of a completely symmetric profile* if each candidate in S appears exactly m times in each position in the profile R_N. Such profiles are important because they allow us to invoke axioms that ensure the impartial treatment of all voters and all candidates; see the properties of anonymity and neutrality introduced below.

A *voting rule* is a mapping $f\colon \mathcal{R} \times \mathcal{X} \to \mathcal{X}$ such that, for all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$, $f(R_N, S) \subseteq S$.

Let $N \in \mathcal{N}$, $R_N \in \mathcal{R}_N$ and $S \in \mathcal{X}$. A candidate $x \in S$ is a *strong Condorcet winner for R_N in S* if

$$|\{i \in N \mid xR_iy\}| > \frac{|N|}{2} \text{ for all } y \in S \setminus \{x\}$$

and $x \in S$ is a *strong Condorcet loser for R_N in S* if

$$|\{i \in N \mid yR_ix\}| > \frac{|N|}{2} \text{ for all } y \in S \setminus \{x\}.$$

Clearly, for any profile R_N and for any feasible set S, there is at most one strong Condorcet winner for R_N in S and at most one strong Condorcet loser for R_N in S. We denote the set of strong Condorcet winners for R_N in S by $CW(R_N, S)$ and the set of strong Condorcet losers for R_N in S by $CL(R_N, S)$. Each of these sets is either empty or a singleton.

A candidate $x \in S$ is a *plurality winner for R_N in S* if

$$|\{i \in N \mid t(R_i, S) = x\}| \geq |\{i \in N \mid t(R_i, S) = y\}| \text{ for all } y \in S.$$

The set of plurality winners for R_N in S is non-empty and may have any number of elements between one and $|S|$. We denote this set by $PW(R_N, S)$. The plurality rule f^p is defined by letting $f^p(R_N, S) = PW(R_N, S)$ for all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$.

A new strong Condorcet loser may appear after the strong Condorcet loser for a profile has been removed. It seems to us that Daunou would have objected to the election of such a candidate and, therefore, we may want to eliminate strong Condorcet losers in a cumulative fashion—that is, after each step in the elimination process, we determine whether there is a candidate that has become a strong Condorcet loser as a consequence of removing those that were disqualified in earlier steps. This procedure can be continued until a set is reached that contains no strong Condorcet loser. This elimination process cannot generate strong Condorcet winners in the reduced sets so that an analogous iterative procedure is not required for strong Condorcet winners.

To show that there are societies $N \in \mathcal{N}$, profiles $R_N \in \mathcal{R}_N$ and feasible sets $S \in \mathcal{X}$ such that a candidate who is not a strong Condorcet loser for R_N in S is both a strong Condorcet loser and a plurality winner for R_N in $S \setminus CL(R_N, S)$, let $N = \{1, 2, 3\}$, $S = X = \{x, y, z, w, v\}$ and define the profile R_N by

$$
\begin{aligned}
&xR_1yR_1zR_1wR_1v,\\
&yR_2zR_2xR_2wR_2v,\\
&wR_3zR_3xR_3yR_3v.
\end{aligned}
$$

It follows that $CW(R_N, S) = \emptyset$, $CL(R_N, S) = \{v\}$, $CL(R_N, S \setminus \{v\}) = \{w\}$ and $w \in PW(R_N, S \setminus \{v\})$. As this example illustrates, it is possible that a new (unique) strong Condorcet loser emerges (w in the example) once the (unique) strong Condorcet loser (in the example, candidate v) is eliminated from a feasible set. It seems to be in the spirit of Daunou's proposal that this new Condorcet loser be removed as well. Of course, this elimination may yield yet another strong Condorcet loser which then is to be removed in another step, and so on.

More formally, we define $CCL(R_N, S)$, the *cumulative set of strong Condorcet losers for R_N in S,* by means of the following iterative method. Let $N \in \mathcal{N}$, $R_N \in \mathcal{R}_N$ and $S \in \mathcal{X}$. If $CL(R_N, S) = \emptyset$, we (trivially) obtain $CCL(R_N, S) = \emptyset$. If $CL(R_N, S) \neq \emptyset$, we define the set $S^1 = S \setminus CL(R_N, S)$ in the first step of the successive elimination process. If $CL(R_N, S^1) \neq \emptyset$, we define $S^2 = S^1 \setminus CL(R_N, S^1)$ and so on until we reach a step K such that no strong Condorcet loser remains—that is, $CL(R_N, S^K) = \emptyset$. Because S is finite, such a step K must exist. The cumulative set of strong Condorcet losers for R_N in S is given by

$$CCL(R_N, S) = CL(R_N, S) \cup CL(R_N, S^1) \cup \ldots \cup CL(R_N, S^{K-1})$$

and, because the last iteratively eliminated strong Condorcet loser must be dominated by *some* candidate(s) in the sense of Condorcet, the set $S \setminus CCL(R_N, S)$ of candidates that remain must be non-empty.

To illustrate, let us return to the above example. Candidate v is the strong Condorcet loser for R_N in $S = \{x, y, z, w, v\}$ so that $CL(R_N, S) = \{v\}$. Thus, v is eliminated and we arrive at the remaining set $S^1 = S \setminus CL(R_N, S) = \{x, y, z, w\}$. In this reduced set, $\{w\} = CL(R_N, S^1) = CL(R_N, \{x, y, z, w\})$ so that w is the strong Condorcet loser for R_N in $\{x, y, z, w\}$. After removing w in this step, the set that remains is $S^2 = S^1 \setminus CL(R_N, S^1) = \{x, y, z\}$. There is no strong Condorcet loser for R_N in $\{x, y, z\}$ so that $CL(R_N, S^2) = \emptyset$ and we obtain the cumulative set of strong Condorcet losers $CCL(R_N, S) = \{w, v\}$. Thus, the set of remaining candidates is given by $S \setminus CCL(R_N, S) = \{x, y, z\}$.

3 Properties of voting rules

The property of *anonymity* guarantees that the rule treats all voters symmetrically, paying no attention to their identities. Let Π be the set of all permutations $\pi\colon \mathbb{N} \to \mathbb{N}$. For all $\pi \in \Pi$ and for all $N \in \mathcal{N}$, let

$$N_\pi = \{j \in \mathbb{N} \mid \exists i \in N \text{ such that } j = \pi(i)\}.$$

Thus, we have

$$R_{N_\pi} = \left(R_{\pi(i)}\right)_{i \in N}$$

for all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $\pi \in \Pi$. Anonymity is now defined as follows.

Anonymity. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$, for all $S \in \mathcal{X}$ and for all $\pi \in \Pi$,

$$f(R_{N_\pi}, S) = f(R_N, S).$$

Our next axiom is *neutrality,* which is the counterpart of anonymity that applies to the equal treatment of the candidates. For a set $S \in \mathcal{X}$, let Σ_S be the set of all permutations $\sigma\colon X \to X$ such that $\sigma(x) = x$ for all $x \in X \setminus S$. For all $i \in \mathbb{N}$, for all $R_i \in \mathcal{R}_i$, for all $S \in \mathcal{X}$ and for all $\sigma \in \Sigma_S$, define the relation $\sigma_i(R_i)$ by letting

$$\sigma(x)\, \sigma_i(R_i)\, \sigma(y) \;\Leftrightarrow\; x R_i y$$

for all $x, y \in X$. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$, for all $S \in \mathcal{X}$ and for all $\sigma \in \Sigma_S$, let

$$\sigma_N(R_N) = \left(\sigma_i(R_i)\right)_{i \in N}.$$

Neutrality. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$, for all $S \in \mathcal{X}$ and for all $\sigma \in \Sigma_S$,

$$f(\sigma_N(R_N), S) = \sigma(f(R_N, S)).$$

The conjunction of anonymity and neutrality implies that, for any profile R_N and any set of candidates S so that R_N is a replica of a completely symmetric profile, the voting rule must choose all elements in S. This is because, in an m-fold replica of a completely symmetric profile, each candidate appears exactly m times in each position. Thus, the equal treatment of the voters (imposed by anonymity) or the equal treatment of candidates (guaranteed by the neutrality axiom) demands that all candidates in S are elected: because the set of chosen candidates is non-empty, the absence of one of the candidates in the selected set would necessarily involve an unequal treatment of the voters or the candidates (or both).

We now address the treatment of strong Condorcet winners and strong Condorcet losers. The *Condorcet winner criterion* guarantees the choice of a strong Condorcet winner whenever such a candidate exists.

Condorcet winner criterion. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$, if $CW(R_N, S) \neq \emptyset$, then $f(R_N, S) = CW(R_N, S)$.

To deal with strong Condorcet losers, we employ a property that differs from the standard Condorcet loser criterion in two respects. As already discussed, we require the conclusion of the axiom to apply to the cumulative set of strong Condorcet losers rather than merely the strong Condorcet loser. In addition, we demand that removing the cumulative set of strong Condorcet losers does not change the selected candidates; this is a strengthening of the *cumulative Condorcet loser criterion* which merely requires that cumulative strong Condorcet losers not be selected. This results in the following axiom of *cumulative Condorcet loser independence.*

Cumulative Condorcet loser independence. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$, $f(R_N, S) = f(R_N, S \setminus CCL(R_N, S))$.

The cumulative Condorcet loser criterion alluded to above is not sufficient for our purposes—it permits the selection from a feasible set to change if a cumulative strong Condorcet loser is removed from this set, as long as a cumulative strong Condorcet loser is not chosen. To illustrate that such a situation can indeed occur, we provide an example in Part A of the Appendix. We note that this observation also applies if merely the strong Condorcet loser rather than the entire cumulative set of strong Condorcet losers is to be removed. Thus, the full force of cumulative Condorcet loser independence (as opposed to the weaker cumulative Condorcet loser criterion) is required in our characterization result.

The following two properties are conditional on the absence of strong Condorcet winners and cumulative strong Condorcet losers. The first of these ensures that, conditional on the absence of strong Condorcet winners and cumulative strong Condorcet losers, a tops-only property is satisfied. This property is familiar from the literature on *single-peaked* preferences; see, for instance, Duncan Black (1958), Michael Dummett and Robin Farquharson (1961) and Hervé Moulin (1980). While the property appears to be quite forceful in the present context, its presence is not too surprising owing to the nature of the plurality rule.

Conditional tops only. For all $N \in \mathcal{N}$, for all $R_N, R'_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$, if $CW(R_N, S) = CW(R'_N, S) = CCL(R_N, S) = CCL(R'_N, S) = \emptyset$ and $t(R_i, S) = t(R'_i, S)$ for all $i \in N$, then

$$f(R_N, S) = f(R'_N, S).$$

The final property in our axiomatization applies to reductions of a profile that involve the removal of voters whose top candidates were chosen prior to their departure. The axiom only applies if the candidate in question is not the only chosen candidate for the original (pre-reduction) profile; this is essential to ensure that the condition is well-defined.

Conditional reduction monotonicity. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$, for all $M \subsetneq N$, for all $S \in \mathcal{X}$ and for all $x \in S$, if $CW(R_N, S) = CW(R_{N \setminus M}, S) = CCL(R_N, S) = CCL(R_{N \setminus M}, S) = \emptyset$, $\{x\} \subsetneq f(R_N, S)$ and $t(R_i, S) = x$ for all $i \in M$, then

$$f(R_{N \setminus M}, S) = f(R_N, S) \setminus \{x\}.$$

Because of their focus on top candidates, the two conditional properties defined above exclude other voting rules that satisfy the Condorcet winner criterion.

4 A characterization of Daunou's voting method

Daunou's proposal can be formalized by means of the following voting rule f^D. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$,

(i) if $CW(R_N, S) \neq \emptyset$, then $f^D(R_N, S) = CW(R_N, S)$;

(ii) if $CW(R_N, S) = \emptyset$, then $f^D(R_N, S) = PW(R_N, S \setminus CCL(R_N, S))$.

Our main result characterizes this voting rule. The independence of the axioms used in this axiomatization is established in Part B of the Appendix. To make the argument employed in the proof of the only-if part of this theorem easier to follow, we provide an informal explanation of the proof structure and an illustrative example in Part C.

Theorem 1 *A voting rule f satisfies anonymity, neutrality, the Condorcet winner criterion, cumulative Condorcet loser independence, conditional tops only and conditional reduction monotonicity if and only if $f = f^D$.*

Proof. That f^D satisfies the axioms of the theorem statement is straightforward to verify. Conversely, suppose that f is a voting rule that satisfies the axioms. Let $N \in \mathcal{N}$, $R_N \in \mathcal{R}_N$ and $S \in \mathcal{X}$. We have to show that $f(R_N, S) = f^D(R_N, S)$. This follows trivially if S is a singleton so we now assume that S contains at least two elements.

If $CW(R_N, S) \neq \emptyset$, it follows that $f(R_N, S) = CW(R_N, S) = f^D(R_N, S)$ by the Condorcet winner criterion.

If $CW(R_N, S) = \emptyset$ we can, without loss of generality, assume that $CCL(R_N, S) = \emptyset$ because f satisfies cumulative Condorcet loser independence. Let

$$n^* = \max_{x \in S} \{ |\{ i \in N \mid t(R_i, S) = x \}| \}$$

denote the plurality-winner score for R_N in S.

Suppose first that $PW(R_N, S) = S$. Let $\overline{R}_N$ be a profile such that $t(\overline{R}_i, S) = t(R_i, S)$ for all $i \in N$ and, moreover, each candidate in $PW(R_N, S) = S$ appears in each position n^* times in $\overline{R}_N$. Thus, $\overline{R}_N$ is an n^*-fold replica of a completely symmetric profile. By anonymity and neutrality, it follows that $f(\overline{R}_N, S) = S = PW(R_N, S)$. We have that $CW(R_N, S) = CCL(R_N, S) = \emptyset$ by assumption, and $CW(\overline{R}_N, S) = CCL(\overline{R}_N, S) = \emptyset$ follows from the definition of $\overline{R}_N$. We can therefore apply conditional tops only to obtain

$$f(R_N, S) = f(\overline{R}_N, S) = S = PW(R_N, S) = f^D(R_N, S).$$

Finally, suppose that $PW(R_N, S) \subsetneq S$. Without loss of generality, let $PW(R_N, S) = \{x_1, \ldots, x_w\}$ and $S \setminus PW(R_N, S) = \{x_{w+1}, \ldots, x_{w+p}\}$ where $w \in \{1, \ldots, |S| - 1\}$ and $w + p = |S|$. Define, for all $k \in \{1, \ldots, |S|\}$,

$$n_k = |\{ i \in N \mid t(R_i, S) = x_k \}|.$$

Clearly, $n_k = n^*$ for all $k \in \{1, \dots, w\}$ and $n_k < n^*$ for all $k \in \{w+1, \dots, w+p\}$.

In the next step of the proof, we construct a larger profile that is an n^*-fold replica of a completely symmetric profile and invoke the conjunction of anonymity and neutrality to conclude that all candidates in S must be chosen. Clearly, we need to add as many voters as required to arrive at an augmented profile so that each candidate in S appears n^* times in the top position. Because this is already the case for the w candidates in $PW(R_N, S) = \{x_1, \dots, x_w\}$, we only need to take care of the p candidates in $S \setminus PW(R_N, S) = \{x_{w+1}, \dots, x_{w+p}\}$. Let the set $N' \subsetneq \mathbb{N} \setminus N$ be composed of $n' = |N'|$ voters, where

$$n' = \sum_{k=w+1}^{w+p} (n^* - n_k).$$

Now consider a profile $\overline{R}_{N\cup N'} = (\overline{R}_N, \overline{R}_{N'})$ such that

$$t(\overline{R}_i, S) = t(R_i, S) \text{ for all } i \in N \tag{1}$$

and

$$|\{i \in N' \mid t(\overline{R}_i, S) = x_k\}| = n^* - n_k \text{ for all } k \in \{w+1, \dots, w+p\} \tag{2}$$

and, moreover, each candidate in S appears n^* times in each position in the profile $\overline{R}_{N\cup N'}$. By (1) and (2), it follows that

$$|\{i \in N \cup N' \mid t(\overline{R}_i, S) = x_k\}| = n^*$$

for all $k \in \{1, \dots, |S|\} = \{1, \dots, w+p\}$. By definition, $CW(\overline{R}_{N\cup N'}, S) = CCL(\overline{R}_{N\cup N'}, S) = \emptyset$. The profile $\overline{R}_{N\cup N'}$ is an n^*-fold replica of a completely symmetric profile and, by anonymity and neutrality, it follows that

$$f(\overline{R}_{N\cup N'}, S) = S.$$

Next, we successively reduce the profile $\overline{R}_{N\cup N'} = (\overline{R}_N, \overline{R}_{N'})$ to the profile $\overline{R}_N$ and apply conditional reduction monotonicity in each step. To ensure that the axiom can indeed be applied, we first show that the elimination of any subset of voters in N' does not create a strong Condorcet winner or a strong Condorcet loser (and, therefore, does not create any cumulative strong Condorcet losers). To do so, let M be any non-empty subset of N'. Now observe that, for any $k \in \{1, \dots, w_{|S|} - 1\}$, there are at least n^* voters corresponding to the profile $\overline{R}_{N\cup(N'\setminus M)}$ who rank x_k above x_{k+1}, and at most n^* voters who rank x_{k+1} above x_k. Therefore, x_k cannot be a (cumulative) strong Condorcet loser and x_{k+1} cannot be a strong Condorcet winner for $\overline{R}_{N\cup(N'\setminus M)}$ in S. Likewise, there are at least n^* voters associated with the profile $\overline{R}_{N\cup(N'\setminus M)}$ who rank $x_{|S|}$ above x_1 so that $x_{|S|}$ cannot be a (cumulative) strong Condorcet loser and x_1 cannot be a strong Condorcet winner. Therefore, all the reduced profiles employed in the subsequent construction are such that there are no strong Condorcet winners and no cumulative strong Condorcet losers so that conditional reduction monotonicity can be applied.

Consider the set $M_1 = \{i \in N' \mid t(R'_i, S) = x_{w+1}\}$. By conditional reduction monotonicity, it follows that

$$f(\overline{R}_{N\cup(N'\setminus M_1)}, S) = f(\overline{R}_{N\cup N'}, S) \setminus \{x_{w+1}\} = S \setminus \{x_{w+1}\}.$$

If $M_1 \subsetneq N'$, this procedure can be used repeatedly where, in each step $q \in \{2, \ldots, p\}$, we define $M_q = \{i \in N' \mid t(R'_i, S) = x_{w+q}\}$ and invoke conditional reduction monotonicity to conclude that

$$f(\overline{R}_{N \cup (N' \setminus (M_1 \cup \ldots \cup M_q))}, S) = f(\overline{R}_{N \cup (N' \setminus (M_1 \cup \ldots \cup M_{q-1}))}, S) \setminus \{x_{w+q}\} = S \setminus \{x_{w+1}, \ldots, x_{w+q-1}, x_{w+q}\}.$$

At the final step $q = p$, we obtain $N' = M_1 \cup \ldots \cup M_p$ and hence

$$\begin{aligned} f(\overline{R}_N, S) &= f(\overline{R}_{N \cup (N' \setminus (M_1 \cup \ldots \cup M_p))}, S) \\ &= f(\overline{R}_{N \cup N'}, S) \setminus \{x_{w+1}, \ldots, x_{w+p}\} \\ &= S \setminus (S \setminus PW(R_N, S)) \\ &= PW(R_N, S). \end{aligned}$$

By conditional tops only (which can be applied because $CW(\overline{R}_N, S) = CCL(\overline{R}_N, S) = \emptyset$), it follows that

$$f(R_N, S) = f(\overline{R}_N, S) = PW(R_N, S) = f^D(R_N, S)$$

and the proof is complete. ■

Our characterization result can be modified easily if merely the strong Condorcet loser rather than the set of cumulative Condorcet losers is to be eliminated. In that case, the set $CCL(R_N, S)$ has to be replaced with $CL(R_N, S)$ in the axioms of cumulative Condorcet loser independence, conditional tops only and conditional reduction monotonicity. It is straightforward to verify that all arguments employed in the above proof remain valid for this alternative specification.

If conditional tops only and conditional reduction monotonicity are amended by removing the requirement that the sets of strong Condorcet winners and the cumulative set of strong Condorcet losers be empty, the resulting unconditional axioms in conjunction with anonymity and neutrality characterize the plurality rule.

A brief comparison with some earlier results on the plurality rule is in order. It appears difficult to adapt Ching's (1996) axiomatization of the plurality rule (which is a strengthening of a characterization by Jeffrey Richelson, 1978) to our conditional setting. Ching's (1996, pp. 299–301) first step consists of showing that if all top elements in a profile are distinct, then all of them must be selected by the voting rule. To establish this claim, he proceeds by induction on the number of voters. While the first step involving a single voter does not create any difficulty in our setting, the induction step cannot be reproduced in our conditional setting. Even if we were to restrict attention to profiles in which there are no strong Condorcet winners and no (cumulative) strong Condorcet losers, the profile with one less voter that is invoked may very well contain a strong Condorcet winner or a strong Condorcet loser. In such a situation, we cannot proceed on the basis of our axioms that only apply when such candidates are absent. Thus, our axiomatization cannot but be different from that of Ching (1996).

Let us conclude this paper with a qualification. Our interest in Daunou's voting rule does not mean that we endorse it as the best method in any sense of the word, even though we surely believe that it has some merit. This belief seems to be well-warranted by our identification of the precise nature of his method as well as our axiomatic characterization thereof.

Appendix

Part A: Insufficiency of the cumulative Condorcet loser criterion. The formal definition of the cumulative Condorcet loser criterion is as follows.

Cumulative Condorcet loser criterion. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$, for all $S \in \mathcal{X}$ and for all $x \in S$, if $x \in CCL(R_N, S)$, then $x \notin f(R_N, S)$.

To define our example of a voting rule f that illustrates the insufficiency of this axiom in our axiomatization, we require the notion of a weak Condorcet loser. Let $N \in \mathcal{N}$, $R_N \in \mathcal{R}_N$ and $S \in \mathcal{X}$. A candidate $x \in S$ is a *weak Condorcet loser for* R_N *in* S if

$$|\{i \in N \mid yR_ix\}| \geq \frac{|N|}{2} \text{ for all } y \in S \setminus \{x\}.$$

Note that it is possible for there to be multiple weak Condorcet losers. The set of all weak Condorcet losers for R_N in S is denoted by $WCL(R_N, S)$.

Consider any population N, any profile R_N and any feasible set S. We construct the set of candidates selected by f according to the procedure described below.

(i) In analogy with the first case in the definition of Daunou's rule, if there is a strong Condorcet winner for R_N in S, select this candidate and only this candidate.

(ii) If there is no strong Condorcet winner for R_N in S and there is no cumulative strong Condorcet loser for R_N in S, all plurality winners for R_N in the set S are selected.

(iii) If there is no strong Condorcet winner for R_N in S and there is a cumulative strong Condorcet loser for R_N in S and the set of plurality winners for R_N in $S \setminus CCL(R_N, S)$ contains at least one candidate who is not a weak Condorcet loser for R_N in $S \setminus CCL(R_N, S)$, the set of plurality winners for R_N in $S \setminus CCL(R_N, S)$ who are not weak Condorcet losers is selected.

(iv) If there is no strong Condorcet winner for R_N in S and there is a cumulative strong Condorcet loser for R_N in S and the set of plurality winners for R_N in $S \setminus CCL(R_N, S)$ only contains weak Condorcet losers for R_N in $S \setminus CCL(R_N, S)$, the set of plurality winners for R_N in $S \setminus CCL(R_N, S)$ is selected.

The above four cases (i) to (iv) can be defined more formally as follows. Let $N \in \mathcal{N}$, $R_N \in \mathcal{R}_N$ and $S \in \mathcal{X}$, and define $f(R_N, S)$ as follows.

(i) If $CW(R_N, S) \neq \emptyset$, then $f(R_N, S) = CW(R_N, S)$;

(ii) if $CW(R_N, S) = CCL(R_N, S) = \emptyset$, then $f(R_N, S) = PW(R_N, S)$;

(iii) if $CW(R_N, S) = \emptyset$ and $CCL(R_N, S) \neq \emptyset$ and

$$PW(R_N, S \setminus CCL(R_N, S)) \setminus WCL(R_N, S \setminus CCL(R_N, S)) \neq \emptyset,$$

then $f(R_N, S) = PW(R_N, S \setminus CCL(R_N, S)) \setminus WCL(R_N, S \setminus CCL(R_N, S))$;

(iv) if $CW(R_N, S) = \emptyset$ and $CCL(R_N, S) \neq \emptyset$ and

$$PW(R_N, S \setminus CCL(R_N, S)) \setminus WCL(R_N, S \setminus CCL(R_N, S)) = \emptyset,$$

then $f(R_N, S) = PW(R_N, S \setminus CCL(R_N, S))$.

The voting rule f satisfies the cumulative Condorcet loser criterion because a cumulative strong Condorcet loser is never chosen. Also, the rule satisfies the properties of anonymity and neutrality because the labels assigned to the voters and the candidates are irrelevant, and it satisfies the Condorcet winner criterion because a (unique) strong Condorcet winner is always selected; see case (i). Conditional tops only and conditional reduction monotonicity are satisfied because the axioms are silent in the presence of a cumulative strong Condorcet loser, and in case (ii)—the only case that is relevant for the two properties—there cannot be a violation because the plurality rule applies in all requisite instances. That cumulative Condorcet loser independence is violated can be seen by examining the following example. Let $N = \{1, 2, 3, 4\}$, $S = X = \{x, y, z, w, v\}$ and consider the profile R_N given by

$$\begin{aligned} &x R_1 w R_1 z R_1 y R_1 v, \\ &y R_2 z R_2 w R_2 x R_2 v, \\ &z R_3 x R_3 y R_3 w R_3 v, \\ &w R_4 x R_4 z R_4 y R_4 v. \end{aligned}$$

There is no strong Condorcet winner for R_N in S, candidate v is the unique strong Condorcet loser for R_N in S, and candidates y and w are weak Condorcet losers for R_N in $S \setminus CCL(R_N, S) = \{x, y, z, w\}$. The set of plurality winners in $S \setminus CCL(R_N, S)$ is $PW(R_N, S \setminus CCL(R_N, S)) = PW(\{x, y, z, w\}) = \{x, y, z, w\}$. According to the above definition, it follows that

$$f(R_N, S) = f(R_N, \{x, y, z, w, v\}) = \{x, z\}$$

because case (iii) applies, and

$$f(R_N, S \setminus CCL(R_N, S)) = f(\{x, y, z, w\}) = \{x, y, z, w\}$$

because case (ii) applies. Clearly, $f(R_N, S) \neq f(R_N, S \setminus CCL(R_N, S))$ so that cumulative Condorcet loser independence is violated. Note that this example can also be employed if only strong Condorcet losers are to be eliminated but not the entire set of cumulative strong Condorcet losers; this is the case because there is no new strong Condorcet loser once v has been removed.

Part B: Independence of the axioms used in Theorem 1. The following examples establish the independence of the axioms used in our characterization. In each of them, the axiom that is violated is indicated.

Anonymity. Let $N \in \mathcal{N}$, $R_N \in \mathcal{R}_N$ and $S \in \mathcal{X}$. Define the *modified plurality score* $mps(R_N, S; x)$ of $x \in S$ for R_N in S by

$$mps(R_N, S; x) = \begin{cases} |\{i \in N \setminus \{1\} \mid t(R_i, S) = x\}| + 2 & \text{if } 1 \in N; \\ |\{i \in N \mid t(R_i, S) = x\}| & \text{if } 1 \notin N. \end{cases}$$

These scores reflect a special status accorded to voter 1: the top candidate of this voter receives twice the weight of all other voters' top candidates. A candidate $x \in S$ is a *modified plurality winner for R_N in S* if

$$mps(R_N, S; x) \geq mps(R_N, S; y) \text{ for all } y \in S.$$

The set of modified set of plurality winners for R_N is S is denoted by $MPW(R_N, S)$. Now define $f(R_N, S)$ as follows.

(i) If $CW(R_N, S) \neq \emptyset$, then $f(R_N, S) = CW(R_N, S)$;

(ii) if $CW(R_N, S) = \emptyset$, then $f(R_N, S) = MPW(R_N, S \setminus CCL(R_N, S))$.

Because voter 1 has a special status, anonymity is violated. All other axioms are satisfied.

Neutrality. Let $N \in \mathcal{N}$, $R_N \in \mathcal{R}_N$ and $S \in \mathcal{X}$. Suppose that x^* is a fixed candidate in X and that n^* is the plurality-winner score for R_N in $S \setminus CCL(R_N, S)$. Now define $f(R_N, S)$ as follows.

(i) If $CW(R_N, S) \neq \emptyset$, then $f(R_N, S) = CW(R_N, S)$;

(ii) if $CW(R_N, S) = \emptyset$ and

$$x^* \in S \setminus CCL(R_N, S) \quad \text{and} \quad |PW(R_N, S \setminus CCL(R_N, S))| \geq 2$$
$$\text{and} \quad |\{i \in N \mid t(R_N, S) = x^*\}| = n^* - 1, \tag{3}$$

then $f(R_N, S) = PW(R_N, S) \cup \{x^*\}$;

(iii) if $CW(R_N, S) = \emptyset$ and (3) does not apply, then $f(R_N, S) = f^D(R_N, S)$.

Neutrality is violated because the candidate x^* has a special status—under the circumstances of case (ii), x^* is elected even with a top-position score that falls short of the plurality-winner score by one vote. All other axioms are satisfied.

Condorcet winner criterion. The plurality rule f^p violates the Condorcet winner criterion and satisfies all other axioms.

Cumulative Condorcet loser independence. Define the voting rule f as follows. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$,

(i) if $CW(R_N, S) \neq \emptyset$, then $f(R_N, S) = CW(R_N, S)$;

(ii) if $CW(R_N, S) = \emptyset$, then $f(R_N, S) = PW(R_N, S)$.

This voting rule violates cumulative Condorcet loser independence because the cumulative Condorcet losers are not removed before applying the plurality rule in part (ii) and, therefore, they are elected for some profiles. Clearly, all other axioms are satisfied.

Conditional tops only. For $i \in \mathbb{N}$ and $R_i \in \mathcal{R}_i$, let $top(R_i)$ be voter i's top candidate for R_i in X and, if X contains at least two elements, let $sec(R_i)$ be i's second-best candidate in X. For all $N \in \mathcal{N}$ and for all $R_N \in \mathcal{R}_N$, define an ordering $\succsim_{R_N}$ on X as follows. For all $x, y \in X$, $x \succsim_{R_N} y$ if and only if

$$|\{i \in N \mid top(R_i) = x\}| > |\{i \in N \mid top(R_i) = y\}|$$

or

$$|\{i \in N \mid top(R_i) = x\}| = |\{i \in N \mid top(R_i) = y\}| \text{ and}$$
$$|\{i \in N \mid sec(R_i) = x\}| \geq |\{i \in N \mid sec(R_i) = y\}|.$$

Now define, for all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$,

(i) if $CW(R_N, S) \neq \emptyset$, then $f(R_N, S) = CW(R_N, S)$;

(ii) if $CW(R_N, S) = \emptyset$, then $f(R_N, S)$ is the set of best elements in $S \setminus CCL(R_N, S)$ according to the ordering $\succsim_{R_N}$.

This is a lexicographic rule that applies the second-place scores as a tie-breaker if the top scores are the same, in violation of conditional tops-only. Because the top scores have priority, conditional reduction monotonicity is satisfied. That the remaining axioms are satisfied is immediate.

Conditional reduction monotonicity. Define a voting rule f as follows. For all $N \in \mathcal{N}$, for all $R_N \in \mathcal{R}_N$ and for all $S \in \mathcal{X}$,

(i) if $CW(R_N, S) \neq \emptyset$, then $f(R_N, S) = CW(R_N, S)$;

(ii) if $CW(R_N, S) = \emptyset$, then

$$f(R_N, S) = \{x \in S \setminus CCL(R_N, S) \mid \{i \in N \mid t(R_i, S) = x\} \neq \emptyset\}.$$

According to case (ii) of this voting rule, all candidates who appear at least once in a top position are elected. This violates conditional reduction monotonicity because there are profiles such that the elimination of a preference relation with a specific chosen candidate at the top from the profile does not remove this candidate from the chosen set. That the remaining axioms are satisfied is straightforward to verify.

Part C: Illustration of the proof method of Theorem 1. To make the argument employed in the proof of the only-if part of our theorem easier to follow, we provide an informal explanation of the proof structure, along with an illustrative example.

Consider a profile R_N and a feasible set S. First, observe that if there is a strong Condorcet winner, then this candidate must be chosen uniquely; this is an immediate consequence of the Condorcet winner criterion. If there is no strong Condorcet winner, we can, without loss of generality, assume that there are no cumulative strong Condorcet losers; this follows from cumulative Condorcet loser independence. What remains to be shown is that, in the absence of strong Condorcet winners and of cumulative strong Condorcet losers, the set of candidates selected by the voting rule must be given by the set of plurality winners.

If the set of plurality winners coincides with the set of candidates S, it follows that each candidate in S has the same plurality-winner score n^* of top positions for the profile under consideration in the set S. Now an auxiliary profile $\overline{R}_N$ can be defined that preserves the top candidates of all voters in N and, moreover, is such that each candidate in S (and, thus, in the set of plurality winners) appears in each position n^* times so that $\overline{R}_N$ is a replica of a completely symmetric profile. By anonymity and neutrality, all candidates in S (and, thus, all plurality winners) must be chosen for the profile $\overline{R}_N$. Because the top candidates are the same for R_N and for $\overline{R}_N$, conditional tops only implies that these plurality winners must be chosen for R_N as well.

The last (and most subtle) case is obtained if the set of plurality winners is a strict subset of the set of feasible candidates S. Again, let n^* be the plurality-winner score for R_N in S. In contrast to the previous case, now there are candidates in S with a number of top positions assigned to them that is less than n^*. To deal with this situation, we first augment the profile R_N by adding as many voters as required to arrive at a profile that is an n^*-fold replica of a completely symmetric profile; that this augmentation is well-defined is established in the formal proof. Denote this augmented profile by $\overline{R}_{N\cup N'}$, where N' is the set of added voters. By anonymity and neutrality, it follows that all candidates in S must be chosen for the profile $\overline{R}_{N\cup N'}$. We then successively reduce the profile $\overline{R}_{N\cup N'}$ to a profile $\overline{R}_N$ that contains only the original voters in N. Each step in this reduction corresponds to one of the candidates that are in S but not in the set of plurality winners. Applying the axiom of conditional reduction monotonicity in each step, we conclude that, once the profile $\overline{R}_N$ is reached, the only candidates that remain chosen are the plurality winners for R_N in S. By construction, the profile $\overline{R}_N$ is such that all top candidates are identical to those in R_N and, using conditional tops only, we arrive at the desired conclusion that the selected set for R_N is the set of plurality winners in S. We also need to show that, at any stage in the reduction process, the set of strong Condorcet winners and the cumulative set of strong Condorcet losers remain empty; this observation is, of course, essential in order to invoke conditional reduction monotonicity. The following example illustrates the proof method just outlined.

Example 1 Let $N = \{1, 2, 3, 4, 5, 6\}$ and $S = \{x, y, z, w, v\}$, and suppose that the profile R_N is given by

$$\begin{aligned}
&xR_1zR_1yR_1wR_1v,\\
&xR_2zR_2yR_2wR_2v,\\
&yR_3vR_3xR_3wR_3z,\\
&yR_4wR_4vR_4xR_4z,\\
&zR_5vR_5xR_5yR_5w,\\
&wR_6zR_6vR_6xR_6y.
\end{aligned}$$

Clearly, $CW(R_N, S) = CCL(R_N, S) = \emptyset$. We have $PW(R_N, S) = \{x, y\}$, $S \setminus PW(R_N, S) = \{z, w, v\}$, and the plurality-winner score is $n^* = 2$. We now add a set of four voters $N' = \{7, 8, 9, 10\}$ with preferences such that z appears $1 = 2 - 1 = n^* - |\{i \in N \mid t(R_i, S) = z\}|$ times at the top of a preference ordering, w is the top candidate in $1 = n^* - 1$ preferences, and v is the best candidate for $2 = n^* - 0$ voters. The reason why the number of added voters is equal to four is that this choice allows us to obtain a replica of a completely symmetric profile. By construction, in the augmented profile, each of the five candidates in S appears $n^* = 2$ times at the top of a preference ordering. Consider the profile $\overline{R}_{N \cup N'} = (\overline{R}_N, \overline{R}_{N'})$ given by

$$\begin{gathered} x\overline{R}_1 y\overline{R}_1 z\overline{R}_1 w\overline{R}_1 v, \\ x\overline{R}_2 y\overline{R}_2 z\overline{R}_2 w\overline{R}_2 v, \\ y\overline{R}_3 z\overline{R}_3 w\overline{R}_3 v\overline{R}_3 x, \\ y\overline{R}_4 z\overline{R}_4 w\overline{R}_4 v\overline{R}_4 x, \\ z\overline{R}_5 w\overline{R}_5 v\overline{R}_5 x\overline{R}_5 y, \\ w\overline{R}_6 v\overline{R}_6 x\overline{R}_6 y\overline{R}_6 z, \\ z\overline{R}_7 w\overline{R}_7 v\overline{R}_7 x\overline{R}_7 y, \\ w\overline{R}_8 v\overline{R}_8 x\overline{R}_8 y\overline{R}_8 z, \\ v\overline{R}_9 x\overline{R}_9 y\overline{R}_9 z\overline{R}_9 w, \\ v\overline{R}_{10} x\overline{R}_{10} y\overline{R}_{10} z\overline{R}_{10} w. \end{gathered}$$

In the augmented profile $\overline{R}_{N \cup N'}$, every candidate appears $n^* = 2$ times in each position and, therefore, the profile is a two-fold replica of a completely symmetric profile. By anonymity and neutrality, it follows that all five candidates must be chosen so that

$$f(\overline{R}_{N \cup N'}, S) = S = \{x, y, z, w, v\}.$$

We now show that, at each stage of the successive reduction process that leads us back to a profile involving the set of voters N, the set of strong Condorcet winners and the cumulative set of strong Condorcet losers remains empty. This allows us to invoke conditional reduction monotonicity in each iteration. To do so, let M be any non-empty subset of $N' = \{7, 8, 9, 10\}$. Observe that, in the profile $\overline{R}_{N \cup (N' \setminus M)}$, candidate x appears above candidate y at least $n^* = 2$ times and candidate y appears above candidate x at most $n^* = 2$ times. Analogously, y appears at least twice above z and z appears at most twice above y; z appears at least twice above w and w appears at most twice above z; w appears at least twice above v and v appears at most twice above w; and, finally, v appears at least twice above x and x appears at most twice above v. Thus, there are no strong Condorcet winners and no (cumulative) strong Condorcet losers in any subprofile of $\overline{R}_{N \cup N'}$ that contains N. As a consequence, conditional reduction monotonicity can be applied in the following process of successively eliminating the added voters. We perform this reduction one top candidate at a time in order to iteratively reduce the set of voters back to the original set N.

Let $M_1 = \{9, 10\}$. Thus, the profile $\overline{R}_{N \cup (N' \setminus M_1)}$ is given by

$$\begin{gathered} x\overline{R}_1 y\overline{R}_1 z\overline{R}_1 w\overline{R}_1 v, \\ x\overline{R}_2 y\overline{R}_2 z\overline{R}_2 w\overline{R}_2 v, \\ y\overline{R}_3 z\overline{R}_3 w\overline{R}_3 v\overline{R}_3 x, \end{gathered}$$

$$
\begin{array}{c}
y\overline{R}_4 z\overline{R}_4 w\overline{R}_4 v\overline{R}_4 x,\\
z\overline{R}_5 w\overline{R}_5 v\overline{R}_5 x\overline{R}_5 y,\\
w\overline{R}_6 v\overline{R}_6 x\overline{R}_6 y\overline{R}_6 z,\\
z\overline{R}_7 w\overline{R}_7 v\overline{R}_7 x\overline{R}_7 y,\\
w\overline{R}_8 v\overline{R}_8 x\overline{R}_8 y\overline{R}_8 z.
\end{array}
$$

Because $t(\overline{R}_i, S) = v$ for all $i \in M_1$, conditional reduction monotonicity implies

$$f(\overline{R}_{N\cup(N'\setminus M_1)}, S) = f(\overline{R}_{N\cup N'}, S) \setminus \{v\} = \{x, y, z, w, v\} \setminus \{v\} = \{x, y, z, w\}.$$

Now let $M_2 = \{8\}$. The profile $\overline{R}_{N\cup(N'\setminus(M_1\cup M_2))}$ is given by

$$
\begin{array}{c}
x\overline{R}_1 y\overline{R}_1 z\overline{R}_1 w\overline{R}_1 v,\\
x\overline{R}_2 y\overline{R}_2 z\overline{R}_2 w\overline{R}_2 v,\\
y\overline{R}_3 z\overline{R}_3 w\overline{R}_3 v\overline{R}_3 x,\\
y\overline{R}_4 z\overline{R}_4 w\overline{R}_4 v\overline{R}_4 x,\\
z\overline{R}_5 w\overline{R}_5 v\overline{R}_5 x\overline{R}_5 y,\\
w\overline{R}_6 v\overline{R}_6 x\overline{R}_6 y\overline{R}_6 z,\\
z\overline{R}_7 w\overline{R}_7 v\overline{R}_7 x\overline{R}_7 y.
\end{array}
$$

Because $t(\overline{R}_i, S) = w$ for all $i \in M_2$, conditional reduction monotonicity implies

$$f(\overline{R}_{N\cup(N'\setminus(M_1\cup M_2))}, S) = f(\overline{R}_{N\cup(N'\setminus M_1)}, S) \setminus \{w\} = \{x, y, z, w\} \setminus \{w\} = \{x, y, z\}.$$

Finally, let $M_3 = \{7\}$. The profile $\overline{R}_{N\cup(N'\setminus(M_1\cup M_2\cup M_3))} = \overline{R}_N$ is given by

$$
\begin{array}{c}
x\overline{R}_1 y\overline{R}_1 z\overline{R}_1 w\overline{R}_1 v,\\
x\overline{R}_2 y\overline{R}_2 z\overline{R}_2 w\overline{R}_2 v,\\
y\overline{R}_3 z\overline{R}_3 w\overline{R}_3 v\overline{R}_3 x,\\
y\overline{R}_4 z\overline{R}_4 w\overline{R}_4 v\overline{R}_4 x,\\
z\overline{R}_5 w\overline{R}_5 v\overline{R}_5 x\overline{R}_5 y,\\
w\overline{R}_6 v\overline{R}_6 x\overline{R}_6 y\overline{R}_6 z.
\end{array}
$$

Because $t(\overline{R}_i, S) = z$ for all $i \in M_3$, conditional reduction monotonicity implies

$$
\begin{aligned}
f(\overline{R}_N, S) &= f(\overline{R}_{N\cup(N'\setminus(M_1\cup M_2\cup M_3))}, S)\\
&= f(\overline{R}_{N\cup(N'\setminus(M_1\cup M_2))}, S) \setminus \{z\}\\
&= \{x, y, z\} \setminus \{z\} = \{x, y\}\\
&= PW(R_N, S).
\end{aligned}
$$

By conditional tops only (which can be applied because $CW(\overline{R}_N, S) = CCL(\overline{R}_N, S) = \emptyset$), it follows that

$$f(R_N, S) = f(\overline{R}_N, S) = \{x, y\} = PW(R_N, S) = f^D(R_N, S),$$

as desired. ■

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