License Prices for Financially Constrained Firms

Preston McAfee & Roberto Burguet

July, 2005
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by

Roberto Burguet*
Institute for Economic Analysis (CSIC)

and

R. Preston McAfee
California Institute of Technology

Barcelona Economics W n° 224

Abstract: It is often alleged that high auction prices inhibit build-out. We investigate this claim under the extreme case of budget-constrained bidders. Low prices maximize overall the gains from trade. If there are n licenses, the price where the budget constraint just binds maximizes consumer surplus if the elasticity of demand is less than one plus 1/n. If demand is elastic, auctions maximize consumer surplus when build-out expenditure greater than one over the elasticity of demand. This appears to be true for most of the auctions run.

Correspondence to: R. Preston McAfee, 228-77 Caltech, Pasadena, CA 91125, USA, mcafee@caltech.edu.

* Burguet acknowledges financial assistance of Barcelona Economics, CREA.
1. Introduction

It is often alleged that high auction prices for spectrum licenses have inhibited the deployment of the related services, to the detriment of consumers. For example, telecom specialist John Tennant said that the bid on the third generation, or 3G, licenses increased the cost of debt service to the point that the companies could not borrow for infrastructure development, and ultimately accounts for the dramatic drop in share prices of the telecom sector. (McClelland, 2003). Similarly, an EC commission studying 3G services identifies the €110 billion paid for licenses as a major constraint on investment. Several nations, including Finland, France, Ireland and Spain, awarded 3G licenses for free, using what are often called "beauty contests," ostensibly because this would lead to a faster deployment of services (Commission of the EC, 2002).

Textbook economic analysis suggests that license prices are sunk costs by the time investment decisions are made, and thus should have no effect on the deployment of services. Moreover, if profitability varies across countries, one might expect the high profitability countries to attract both high auction prices and rapid and extensive deployment to capture the high profitability. However, a correlation between high auction prices and fast deployment need not disprove the allegation that auction prices inhibit the deployment unless some means of adjusting for profitability is found. Even if profitability is constant, the fallacy of sunk costs suggests, in addition, that psychologically the managers should want to invest more in the regions with high-priced licenses, not less.

On the other hand, starting with Michael Jensen’s 1986 free cash flow concepts, modern corporate finance emphasizes the importance of restraining managers by
limiting their ability to invest. Moral hazard, in the form of career concerns or limited liability, can induce managers to take excess risks. The natural response to such managerial problems is to limit the ability of the manager to make bad choices, either by imposing a budget constraint on the manager, or requiring the manager to use a much higher discount factor than the actual average cost of capital for a project under consideration. Even if budgets are "soft," in the sense that there is always more money, individual executives may bear a career cost of asking for more money, perhaps because they are seen as having mis-estimated the costs, making them hesitate to request more money unless the gains are very large. Such a situation mirrors a budget constraint, at least for some realizations of costs.

The recognition that agency problems -- either moral hazard or asymmetric information or both -- might have an impact on corporate financing and investment probably begins with Stiglitz and Weiss (1981), which argues that asymmetric information can impede credit markets, and Greenwald, Stiglitz and Weiss (1984), which argues that equity financing does not cure the agency problem created by asymmetric information. Another important paper was Myers and Majluf’s 1984 analysis that asymmetric information can drive a wedge between the interests of new investors and creditors, thereby creating an agency problem distinct from that identified by the Stiglitz and Weiss. Lewis and Sappington (1989a, 1989b) show that asymmetric information can lead to inflexible rules, and in particular may fix capital investment at a level unaffected by the state of the world unless the state is very extreme. This works precisely like a budget constraint provided to a manager. Hart and Moore (1995) develop Jensen’s free cash flow concept, and show that debt seniority can be used as a
versatile instrument to induce more efficient project selection. In particular, a mix of "hard" debt, which cannot be postponed, and soft debt create a limit on the ability to raise future capital, thus inducing future budget constraints. Clementi and Hopenhayn (2003) develop a dynamic model in which borrowing constraints arise endogenously and relax as the value of the prospects of the firm improve.

Many empirical tests corroborate the view that firms are budget-constrained to some degree, by showing that internal and external financing are not perfect substitutes. The theme of the empirical studies is that investment decisions are affected by the amount of cash on hand in the firm. Fazzari and Athey (1987) show that the availability of internal financing affects investment. Fazzari, Hubbard and Petersen (1988) emphasize financing constraints in their study of the determinants of investment. Whited (1992) corroborates the existence of financing constraints. Fazzari and Petersen (1993) uses working capital as a way of controlling for errors in measured variables that might create the spurious appearance of financial constraints, and finds evidence that previous studies had in fact underestimated the importance of financing constraints. More recently, Love (2003) estimates the effects of financing constraints across many nations, and finds that strong capital markets in developed nations reduce, but don’t eliminate, the significance of financing constraints. Finally, the empirical literature is also extensively discussed in Clementi and Hopenhayn (2003). If these studies are relevant to the telecom firms, then the cost of spectrum licenses could have an impact on investment in deployment of services.

The behavior of the telecom industry during the 1990s reinforces the importance of these ideas. Some companies bid in excess of the maximum values suggested by
their own analyses. Stefan Zehle describes a 3G bidder in the U.K. who bid £5 billion for a license that the company estimated was worth £1 billion. He also describes an executive who called the auctions a "spectrum landgrab" and that the bidders should not worry whether the prices made business sense (McClelland, 2003). One author (McAfee) was repeatedly asked by spectrum bidders for auction-theoretic reasons for bidding in excess of the net present value. The managers were very disappointed to hear about the winner’s curse, which goes the other direction.¹ In addition, many of the bidders believed that other bidders faced budget constraints and consulted with economists in an attempt to estimate just what those constraints might be.

Ultimately, the use of budgets to restrain managerial excesses makes the allegation that high license prices reduce investments possible. In this paper, we investigate that allegation in an extreme model, in which firms have fixed and "hard" budgets. That is, if license prices are high enough, firms will have less money to invest in the deployment of the services, and the quantity of services offered falls as the license prices rise.

Auctions with budget constraints have been examined by Pitchik and Schotter (19886, 1988), Che and Gale (1988), and Benoit and Krishna (2001). The focus of these papers is on the firms’ ability to bid in subsequent auctions, given the prices paid in earlier auctions, and on the proposition that bidders might artificially inflate the price of earlier sales as a means of reducing the ability of the winners to pay for later items. In contrast, we examine the ability of firms to deploy a service after the sale.

¹ The likely reason for this tendency to bid in excess of net present value was the 1980s experience. The actual number of U.S. cell phone users in 1990 was ten times what was estimated in 1980, and cellular profits represented a large fraction of total telecom profits, mostly because there were only two firms
In a world of financially constrained firms, do auctions maximize consumer surplus, counting the revenue raised by the auction as part of consumer surplus, or do auctions inhibit the delivery of services as the suppliers contend? We characterize conditions under which auctions are optimal. If the firms are not constrained at a zero price, then prices should be increased at least to the lesser of the price at which the firms become constrained and the auction price. If the elasticity demand, $\varepsilon$, is less than $1 + 1/n$, where $n$ is the number of firms, at this price, then this price is the consumer surplus-maximizing price. This price is less than the auction price if financial constraints are relevant. In contrast, if $\varepsilon$ exceeds $1 + 1/n$, then the consumer-surplus maximizing price is the auction price, provided this does not entail spending more than $(\varepsilon-1)/\varepsilon$ of the budget on the licenses. Otherwise, the price that maximizes consumer surplus is between the auction price and the price at which budget constraints just bind.

The formulas have the virtue of being simple and readily checkable. In particular, in the United States PCS auctions, licenses costs were estimated to be about 40% of the costs of deploying a PCS service. This means that, even if the firms were financially constrained, an auction was consumer surplus-maximizing provided the elasticity of demand for PCS services exceeded 1.66. Since the demand for services seems quite elastic, an auction maximized consumer surplus.\textsuperscript{2}

\footnote{in most regions and limited capacity. This dramatic underestimate of the value of wireless fueled an unjustified. For an opposing view of the European 3G auctions, see Klemperer (2002).}

\footnote{Over the rollout of PCS services, prices have dropped by 50% or so, and the number of customers has grown by at least several hundred percent, suggesting elasticities over 4. However, the technology has changed substantially as well, with smaller phones with many more features like cameras and instant messaging, which may account for some of the increased sales.}
2. The Model

There are \( n \) licenses, and at least \( n+1 \) identical firms. A license is a right to compete in a Cournot industry. If industry output is \( Q \), then the realized price is \( p(Q) \). We assume that for all \( Q \),

\[
2p'(Q) + Qp''(Q) < 0.
\]

Inequality (1) is the condition that marginal revenue is downward sloping, and insures that the second order conditions hold globally. For constant elasticity of demand, inequality (1) is equivalent to elastic demand.

Let \( \lambda \) be the price of a license, and \( B \) the budget of each firm. We model the budget constraint as a "hard" budget constraint, primarily to favor the case that budget constraints might interfere with subsequent production. That is, "soft" budget constraints are generally going to have less of an effect than "hard" budget constraints. If a firm chooses to produce the quantity \( q \), the budget constraint becomes

\[
cq + \lambda \leq B,
\]

where \( c \) is the marginal cost of output. We assume that \( c \) is below the demand price \( p(Q) \) for some positive quantity \( Q \).

We look for a symmetric equilibrium in output. Suppose the symmetric equilibrium quantity choice is \( q^* \). Each firm´s profits are

\[
\pi_j = \max_{cq + \lambda \leq B} p(q + (n-1)q^*)q - cq - \lambda.
\]

The elasticity of demand is

\[
\varepsilon(Q) = -\frac{p(Q)}{Qp'(Q)}.
\]
Where the risk of confusion is minimal, we will suppress the dependence of \( \varepsilon \) on \( Q \). We first consider the quantity choices of the firms, which is characterized in Lemma 1.

**Lemma 1:** There is a unique Cournot equilibrium, and it is symmetric and has industry output \( Q \) satisfying:

\[
 p(Q) \left( 1 - \frac{1}{n\varepsilon(Q)} \right) - c \begin{cases} 
 0 & \text{if } cQ \cdot n + \lambda < B \\
 \geq 0 & \text{if } cQ \cdot n + \lambda = B
\end{cases}
\]

All proofs are contained in the appendix.

When the budget constraint does not bind, condition (5) can be expressed

\[
 p(Q) = \frac{n\varepsilon}{n\varepsilon - 1} c ,
\]

and the associated profits are

\[
 \pi = p(Q) \frac{Q}{n} - c \frac{Q}{n} - \lambda = \left( \frac{n\varepsilon}{n\varepsilon - 1} - 1 \right) c \frac{Q}{n} - \lambda = \frac{cQ}{n} - \frac{1}{n\varepsilon - 1} - \lambda .
\]

If the budget constraint binds,

\[
 p \left( n \frac{B - \lambda}{c} \right) \geq \frac{n\varepsilon}{n\varepsilon - 1} c ,
\]

and profits are

\[
 \pi = p \left( n \frac{B - \lambda}{c} \right) \frac{B - \lambda}{c} - c \frac{B - \lambda}{c} - \lambda = p \left( n \frac{B - \lambda}{c} \right) \frac{B - \lambda}{c} - B .
\]

What is the optimal license price \( \lambda \)? We will consider two scenarios, depending on whether the firm profits are counted as part of welfare. If the firms are local firms, and their profits are fully counted as part of the local welfare, then it is appropriate to
maximize the total gains from trade. In this case, giving the licenses away (which relaxes the firms’ budget constraints) maximizes welfare. If, in contrast, the profits of the firms are not part of the objective of the licensor, the results are somewhat more interesting. However, for sufficiently elastic demand, an auction is optimal, and we will derive a quite sharp characterization of how elastic demand must be.

If the licensing authority counts firm profits in welfare, the welfare measure, as a function of output, is composed of three terms, the consumer surplus, the firm profits, and the license revenue. If \( Q \) is the quantity produced by the industry, welfare is:

\[
W = \int_0^Q (p(x) - p(Q))dx + \lambda \left( \frac{Q}{n} p(Q) - c \frac{Q^n}{n} - \lambda \right) + n\lambda = \int_0^Q p(x)dx - cQ = \int_0^Q (p(x) - c)dx.
\]

In this case, the closer is output to the level where price equals marginal cost, the more efficient is the solution. Since price exceeds marginal cost at the unconstrained solution, and price is increasing (weakly if the budget constraint does not bind) in \( \lambda \), \( W \) is maximized at \( \lambda = 0 \), or giving the licenses away free. More generally, any \( \lambda \) small enough that the budget constraint does not bind is optimal. If the budget constraint binds strictly, the value of \( \lambda \) is sub-optimal. Note that this could entail a negative value of \( \lambda \) if the budget constraint binds at \( \lambda = 0 \).

Now consider the consumer surplus, which doesn’t count firm profits, but does count license revenues. This is reasonable for a local government seller when the firms are, for example, from out of the region. The consumer surplus is

\[
CS = \int_0^Q (p(x) - p(Q))dx + n\lambda.
\]
The maximum that any firm would pay for a license is the level leading to zero profits. This is also the level that would be raised by an auction, since at least one firm does not win a license and thus earns zero profits, and bidding by this firm would lead to an auction price inducing zero profits. We denote the zero profits license price by \( \lambda_0 \) and the level that leads to the budget constraint just binding by \( \lambda_B \). Note that \( \lambda_B \) is defined by

\[
\rho \left( \frac{B - \lambda_B}{c} \right) = \frac{n \varepsilon - c}{n \varepsilon - 1}.
\]

The auction price satisfies

\[
\lambda_0 = (p(Q) - c) \frac{Q}{n}
\]

where \( Q \) the market quantity given the auction price, which may either be constrained or unconstrained by the budget.

**Theorem 1:** The value of \( \lambda \) that maximizes CS is at least \( \lambda_B \). If for all \( Q, \varepsilon \leq \frac{n + 1}{n} \), the value of \( \lambda \) that maximizes CS is \( \lambda_B \). If for \( Q = \frac{B - \lambda_B}{c}, \varepsilon > \frac{n + 1}{n} \), the CS-maximizing value of \( \lambda \) exceeds \( \lambda_B \).

Theorem 1 is essentially bad news for auction pricing. It says that with inelastic demand, consumer surplus is maximized with low prices, prices so low that license prices have no effect on investment. If a reduction in price increases investment increases deployment of the service, the price is too high. On the other hand, if demand is more elastic, then the theorem indicates license prices should be high enough to have some effect on investment. However, this theorem is silent on how high
that price should be, and in principle the consumer surplus maximizing price might be substantially lower than the auction price.

When is an auction price optimal? It simplifies the analysis to assume that the elasticity of demand is non-decreasing in Q. This condition is sometimes known as Marshall’s Second Law of Demand (Marshall, 1920), and is satisfied by many familiar functional forms, such as linear and constant elasticity. Given this assumption, we have Theorem 2: Suppose, at the elasticity of demand obtaining at the auction price \( \lambda_0 \), that

\[ \frac{\lambda_0}{B} \leq \frac{\varepsilon - 1}{\varepsilon}, \text{ an auction maximizes CS among all prices with voluntary participation.} \]

Theorem 2 states that if the portion of the total budget spent on the bids is not too large, then an auction maximizes consumer surplus. An attractive feature of the theorem is that it depends on features of the environment which can readily be estimated -- demand elasticities and expenditures on licenses and deployment. A remarkable feature of the theorem is that it does not depend on the number of firms, and in fact obtains even in the case of two firms bidding to be a monopolist. 

Remark: An equivalent version of Theorem 2 is that, for an auction to be optimal, the elasticity should exceed \( \frac{B}{B - \lambda_0} \).

Theorem 2 is exact in the sense that if the inequality is reversed, the auction price is higher than the price which maximizes consumer surplus. Note that the condition automatically fails if the elasticity of demand is less than one. In the next section, we will consider the application of Theorem 2 to the European experience with 3G.
How many firms should obtain licenses? There are two cases, depending on whether the price is less than the auction price or not. If the price is less than the auction price, the budget constraint will bind at the consumer surplus maximizing price, and we can rewrite consumer surplus, using (11) and the budget constraint, as:

\[ CS = \int_0^Q (p(x) - p(Q))dx + nB - cQ. \]

Increasing \( n \) has effects only if it increases the total budget \( nB \) or changes the feasible quantities that are feasible. Increasing the number of firms relaxes the constraint on the feasible set of quantities, so when the price is less than the auction price, it is never desirable to reduce the number of firms, and can be strictly desirable to increase the number.

If the auction price is achieved, we can write consumer surplus as

\[ CS = \int_0^Q (p(x) - c)dx, \]

using (11) and (13). In this case, an increase in the number of firms matters only if it affects quantity, which can happen through an increase in resources if budgets bind, or through an increase in the Cournot quantity when they do not. In sum, more licenses never lower, and may increase, consumer surplus. Moreover, a larger total budget never lowers consumer surplus.

3. The European 3G Experience

Licenses for spectrum intended for 3G (third generation) cellular telephony usage were assigned beginning in March 2000 with Spain. The first auction of 3G licenses
took place in the United Kingdom and ended in April 2000. The four incumbent GSM operators (Vodafone, BT’s O2, France Telecom’s Orange, and T-Mobile) and a new entrant, Hutchison, each won a license. Prices were considered astronomically high, because the prices exceeded the prices for the US PCS spectrum, in spite of the US spectrum having fewer constraints on usage. About the same time as the UK auctions, the stock price index of telecoms started declining (see EC 2002, exhibit 26). By the time the next auction took place in the Netherlands, only three months later, telecom firms had lost about 25% of their equity value. This time, each of the four incumbents (KPN, O2, T-Mobile, and Dutchtone, Orange) won a license. A month later, when the German auction closed, telecom share prices had fallen even further, to about two-thirds of their March 2002 value. In Germany 6 licenses were sold.\(^3\) Again, each of the four incumbents (T-Mobile, O2, Vodafone-, and Mobilcom, which is partly owned by France Telecom) obtained a license, and two new firms, Quam (a joint venture of Telefonica and Sonera) and Orange, entered the market. The next auction took place in Italy, in October 2000. By then, the stock market index of European telecos had already lost more than 40% of its value, as compared to a loss by the American counterparts of about 25% during the same period.

Licenses included obligations to deploy 3G networks with minimum coverage requirements and deadlines. For instance, license holders in the UK were required to have a network in place that covered 80% of the UK population by the end of 2007. In Sweden, the conditions of the beauty contest pushed this to a 99.98 of the population by the end of 2003. In the Netherlands, the requirements included coverage of at least 3

\(^3\) 12 blocks were on sale, and each firm could buy two or three of these blocks. Thus, the number of licenses was endogenously determined.
60% of the population by 2007, and in Germany 25% of the population by the end of 2003 and 50% by the end of 2005.

Immediately after the first wave of license allocations as the year 2000 ended, the mood in the industry changed. As some say, the internet and telecom bubble burst. The prospect of profitable 3G services receded. If only a few months earlier the market was in the peak of the optimism about the telecom industry, by the end of 2000 and beginning of 2001 the articles in the financial press were filled with comments about the struggling of telecom firms with debt crises. The debt taken to finance the acquisition of licenses was often identified as an important contributory factor of the telecom debt crisis. With the equity markets hostile to telecoms, most European telecoms borrowed a substantial amount of money.4

In this landscape of diminished expectations, the launching of 3G services was delayed. In fact, with the unsuccessful exception of Hutchison's 3, the launching of 3G services did not begin until mid-2004. Mobilcom and Quam in Germany and Orange in Sweden had failed to meet their roll-out obligations and consequently had to return their licenses).

In all countries, firms lobbied for breaks in their 3G coverage obligations, and in most places they succeeded. Sweden allowed a year extension on the requirement of (virtually) full population coverage (from the end of 2003 to the end of 2004). Even this extended deadline was not met. In addition, operators received permission to sharing their networks, so that the originally envisioned structure of one independent, competing network per license was lost. Thus, network sharing agreements among carriers were approved by national governments and went unopposed across Europe, including the
UK, Sweden, Germany, and the Netherlands. As of February 2005, population coverage of 3G networks had reached only 85% in Sweden, 75% in the UK, and less than 60% in the rest of Europe.

The demand elasticity is a critical input to the theory. Earlier studies in wireless telephony obtain elasticity estimates in the range 0.50-1.0.\textsuperscript{5} However, early adopters of cellular telephony probably had relatively inelastic demand, so that demand at lower prices may be substantially more elastic than these estimates suggest. In addition, it is also widely thought that the demand for 3G services such as video and gaming is even more elastic than wireless telephony. Wallenius and Hämäläinen (2002) estimate the elasticity of demand for 3G services to be in the range 1.4-1.7, although their source is not identified.

If the demand elasticity were 1.5, then auction prices would maximize consumer surplus if license prices accounted for less than a third of the firms' budgets, even if the firms were budgets constrained. Given the problems that telecom firms faced with borrowing in the 2001-2005 period, it seems likely that the firms were constrained.

A proxy for the firms' budgets is the sum of estimated cost of deploying a 3G network plus the license fee.\textsuperscript{6} Taking Western Europe as a whole\textsuperscript{7}, the total cost of building networks for all licensees (in the 2000-01 sales) has been estimated at 140B €, whereas total cost of licenses was 120B €,\textsuperscript{8} a ratio of license to total cost of almost $\frac{1}{2}$. However, most of the cost of licenses is accounted for by the British and German

\textsuperscript{4} The Economist, January 25, 2001.
\textsuperscript{5} See, for instance Rodini et al. (2002) or Hausman (1999), (2000).
\textsuperscript{6} This figure ignores marketing and other costs of operating a network, but also ignores network sharing. As it treats the cost of building a network as fixed, it tends to over-estimate the budget.
\textsuperscript{7} All figures used in this paragraph are taken from EC (2002)
auctions, which raised total of 86B €. In the UK, license prices total 36B €, compared to an estimated 21B € needed in network investment. License fees appear close to two-thirds of the total cost (license plus network) of deploying 3G services. Similarly, in Germany the license cost was 50B € and estimated cost of the network only 34B €, so that license fee accounted for 60% of deployment cost. In the rest of the countries that used auctions to assign licenses, the ratio of license fees to total estimated costs ranged from 12% in Greece to 34% in the Netherlands.

The remark in the previous section permits calculating elasticity necessary to justify auction prices, assuming the budget constraints bind. Auction prices maximized consumer surplus in the UK only if the elasticity of demand exceeded 2.7. In Germany, the critical elasticity is about 2.5. These elasticities exceed most estimates for the elasticity of demand for 3G services, suggesting that the prices indeed were too high to maximize consumer surplus, unless budget constraints for the firms did not bind. In other countries, however, the critical elasticity is 1.5 or less, suggesting that an auction maximized consumer surplus.

Were the firms actually budget constrained? Budget constraints tend to create a negative correlation between license prices and build-out. In Figure 1, we graph the build-out, in percentage, against per capita license costs. A weak positive correlation is present, which does not corroborate the conclusion of budget constraints. Higher auction prices are associated with more extensive build-out. However, Figure 1 does

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8 Our source is EC (2002). The figures refer to 3G network deployment, and do not include upgrades and replacement investment for 2G networks. The estimated total investment in this category needed for the period 2000-2015 is 90B €.

9 These numbers are corroborated by the experience of O2, the originally BT mobile company. It is estimated that it spent a total 4B £ (approx. 6B €) in building its 3G networks, mainly in the UK and Germany. It spent around 15B € acquiring its British and German licenses. License fees in both markets represented more than 70% of its estimated budget (network cost plus license fees) for both markets.
not hold the overall market profitability constant, and it could be that the overall size of the budget constraint is correlated with factors like consumer income, which would induce a positive correlation. Thus, while Figure 1 offers no support for budget constraints, it also does not disprove budget constraints. Germany and the UK are the two extreme data points.

![Figure 1: The approximate buildout percentage versus per capita license cost in dollars, European 3G licenses.](image)

3. Conclusion

Both experience with telecommunications companies and corporate finance research indicates that budget constraints are a fact of life in many bidding contexts. In principle, this means that company complaints that high auction prices prevented the rapid rollout of services could have merit. The effect of budget constraints on the deployment of services was examined in the context of a simple model of budget constraints. We found that the evaluation of the effect hinges on relatively inelastic
demand, and that auctions are optimal even when the firms are budget constrained, provided the auction price isn’t too large a fraction of the firms’ resources.

As a practical matter, it appears that in most countries, the auction price was not too high, and thus that auctions were the best way to allocate the licenses. Only in the U.K. and possibly Germany was the price so high as to potentially constraint the rollout of services.

Auctions have an additional advantage obscured by the symmetry of the model: auctions select the efficient service providers. Even if demand is relatively inelastic, it may be desirable to auction in order to select efficiently. However, in such a setting, auctions could have a perverse effect if the most efficient firms have relatively small budgets, because auctions favor both the efficiency and large budgets. Nevertheless, we expect that the advantages of auctions over random selection are greater when firms are differentiated than in our simple model.
References


Pitchik, Carolyn, and Andrew Schotter, "Budget Constrained Sequential Auctions, 1986, unpublished manuscript.


Appendix

Proof of Lemma 1: For the moment, ignore the budget constraint. Fix the output of other firms at $Q_{-i}$, so that profits are

$$\pi = p(q + Q_{-i})q - cq - \lambda.$$  

If the constraint does not bind, the second derivative of profits is

$$\frac{\partial^2 \pi}{(\partial q)^2} = qp''(q + Q_{-i}) + 2p'(q + Q_{-i}).$$

If $p''(q + Q_{-i}) < 0$, $\frac{\partial^2 \pi}{(\partial q)^2} < 0$ since $p$ is a demand curve. If $p''(q + Q_{-i}) > 0$,

$$\frac{\partial^2 \pi}{(\partial q)^2} = qp''(q + Q_{-i}) + 2p'(q + Q_{-i}) < (q + Q_{-i})p''(q + Q_{-i}) + 2p'(q + Q_{-i}) < 0$$ by (1).

Either way, $\pi$ is globally concave, so the Kuhn-Tucker condition characterizes a maximum.

The Kuhn-Tucker condition is

$$\frac{\partial \pi}{\partial q} = qp'(q + Q_{-i}) + p(q + Q_{-i}) - c \begin{cases} = 0 & \text{if } cq + \lambda < B \\ \geq 0 & \text{if } cq + \lambda = B \end{cases}$$

or

$$qp'(Q) + p(Q) - c \begin{cases} = 0 & \text{if } cq + \lambda < B \\ \geq 0 & \text{if } cq + \lambda = B \end{cases}$$

or

$$p(Q) \left(1 - \frac{q}{Q} \frac{1}{c'(Q)}\right) - c \begin{cases} = 0 & \text{if } cq + \lambda < B \\ \geq 0 & \text{if } cq + \lambda = B \end{cases}$$
Note that, if firm \( i \) is constrained, then any firm producing the quantity \( q \) less than \( q_i \) satisfies
\[
p(Q) \left( 1 - \frac{q}{Q} \frac{1}{c(Q)} \right) - c > p(Q) \left( 1 - \frac{q_i}{Q} \frac{1}{c(Q)} \right) - c \geq 0.
\]

Thus, if one firm is constrained, they are all constrained. That is, either no firm, or all firms, are constrained. Consequently, (1) entails that any equilibrium is symmetric, and satisfies
\[
p(Q) \left( 1 - \frac{1}{n c(Q)} \right) - c \begin{cases} = 0 & \text{if } cQ/n + \lambda < B \\ \geq 0 & \text{if } cQ/n + \lambda = B \\ \end{cases}.
\]

Note that
\[
\frac{d}{dQ} p(Q) \left( 1 - \frac{1}{n c(Q)} \right) - c = \frac{1}{n} \frac{d}{dQ} \left( np(Q) + Qp'(Q) \right) = \frac{1}{n} \left( (n + 1)p'(Q) + Qp''(Q) \right)
\]
\[
= \frac{1}{n} \left( (n - 1)p'(Q) + 2p'(Q) + Qp''(Q) \right) \leq \frac{1}{n} \left( 2p'(Q) + Qp''(Q) \right) \leq 0, \text{ by (1)}.
\]

Thus, there is a unique Cournot equilibrium.

**Proof of Theorem 1:**

If the budget constraint doesn’t bind, the quantity is locally independent of \( \lambda \).

Then,
\[
\frac{dCS}{d\lambda} = n > 0. \text{ Thus it increases CS to increase } \lambda \text{ at least to the point that the budget constraint binds, demonstrating the CS-maximizing } \lambda \text{ is at least } \lambda_B.
\]

If, instead, the budget constraint binds, the quantity is
\[
Q = n \frac{B - \lambda}{c}.
\]
\[
CS = \int_0^Q (p(x) - p(Q))dx + n\lambda = \int_0^Q (p(x) - p(Q))dx + nB - cQ. \text{ Thus,}
\]
\[
\frac{dCS}{d\lambda} = -\frac{n}{c} \frac{dCS}{dQ} = -\frac{n}{c} (-Qp'(Q) - c) = -\frac{n}{c} \left( \frac{p(Q)}{\epsilon} - c \right) = \frac{n}{\epsilon c} (\epsilon c - p(Q)).
\]

When the budget binds, it is CS-maximizing to increase \( \lambda \) when \( p \left( \frac{nB - \lambda}{c} \right) \leq \epsilon c \).

Suppose \( \epsilon \leq \frac{n+1}{n} \). For \( \lambda \geq \lambda_B \), \( p \left( \frac{nB - \lambda_B}{c} \right) \geq \frac{nc}{n\epsilon - 1} \). In addition that \( \epsilon \leq \frac{n+1}{n} \) implies \( n\epsilon - 1 \leq n \), or \( 1 \leq \frac{n}{n\epsilon - 1} \), and thus that \( \epsilon c \leq \frac{nc}{n\epsilon - 1} \leq p \left( \frac{nB - \lambda_B}{c} \right) \), and hence the CS maximizing value of \( \lambda \) is never greater than \( \lambda_B \).

Now suppose \( \epsilon > \frac{n+1}{n} \). At \( \lambda_B \), \( p \left( \frac{nB - \lambda_B}{c} \right) = \frac{nc}{n\epsilon - 1} \). In addition that \( \epsilon > \frac{n+1}{n} \) implies \( n\epsilon - 1 > n \), or \( 1 > \frac{n}{n\epsilon - 1} \), and thus that \( \epsilon c > \frac{nc}{n\epsilon - 1} = p \left( \frac{nB - \lambda_B}{c} \right) \), and hence the CS maximizing value of \( \lambda \) exceeds \( \lambda_B \).

**Proof of Theorem 2:**

First note that in the proof of theorem 1, it was demonstrated that it is profitable to increase \( \lambda \) whenever \( p \left( \frac{nB - \lambda}{c} \right) \leq \epsilon c \). If budget constraint does not bind at the auction price, we know from theorem 1 that it increases CS to increase the price to the auction price. Thus, for the remainder of the proof, we assume the budget constraint binds at the auction price.
Marshall’s second law makes $p(Q) - \varepsilon c$ a decreasing function of $Q$, and thus

insures that if $p\left( n \frac{B - \lambda_0}{c} \right) \leq \varepsilon c$, then $p\left( n \frac{B - \lambda}{c} \right) \leq \varepsilon c$ for all $\lambda \leq \lambda_0$. Thus, we need only check when $p\left( n \frac{B - \lambda_0}{c} \right) \leq \varepsilon c$. But at $\lambda_0$,

$$0 = p\left( n \frac{B - \lambda_0}{c} \right) (B - \lambda_0) - B = \left( p\left( n \frac{B - \lambda_0}{c} \right) - c \right) \left( \frac{B - \lambda_0}{c} \right) - \lambda_0.$$

Thus

$$p\left( n \frac{B - \lambda_0}{c} \right) \leq \varepsilon c$$

if and only if

$$p\left( n \frac{B - \lambda_0}{c} \right) - c \leq (\varepsilon - 1)c$$

if and only if

$$\lambda_0 = \left( p\left( n \frac{B - \lambda_0}{c} \right) - c \right) \frac{B - \lambda_0}{c} \leq (\varepsilon - 1) \frac{B - \lambda_0}{c} = (\varepsilon - 1)(B - \lambda_0)$$

if and only if $\varepsilon \lambda_0 \leq (\varepsilon - 1)B$ if and only if $\frac{\lambda_0}{B} \leq \frac{\varepsilon - 1}{\varepsilon}$.