The Condominium Problem; Auctions for Substitutes

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Abstract

This paper considers the problem of designing selling procedures for substitutes (like condominium units). I show that oral, ascending auctions for the right to choose are efficient. This is a common type of auction used for the sale of real estate. Efficiency is not optimal from the seller’s viewpoint. An optimal procedure distorts the right-to-choose auction to favor in late rounds bidders whose preferred object has already been sold. This optimal auction is complex. A revenue improving departure from efficiency can be achieved by simply auctioning all the rights to choose before any of them is exerted. This is also a common feature of auctions for the sale of real estate.

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1 Introduction

This paper studies the problem of selling procedures for substitutes, as illustrated by the sale of condominium units. Assume you own a number of apartments in some apartment building that you are considering selling. Auction theory has treated with some depth the sale of several units of homogeneous goods to buyers who are willing to acquire at most one of them. Even though more incomplete, the analysis has also been extended to situations where buyers are willing to buy more than one unit. Yet, apartments are not "exactly" homogeneous. They may be "substitutes" but they are not "perfect substitutes". Some face west, some face east. Some receive more sunshine but are noisier too. In fact, no two apartments are identical. Not only are they different, but it may difficult to decide ex-ante which one is "better". Different buyers may be expected to give different answers to that question.

Auction theory offers little guidance for this problem, which I term "the condominium problem". That is, the sale of several objects with the following features:

- Buyers have different willingness to pay for different objects: objects are distinct.

- The willingness to pay for one of the objects depends on whether the buyer already owns another one. In particular, owning one object reduces the willingness to pay for another one, perhaps to zero.

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1 Bernhardt and Scoones (1994), and Gale and Hausch (1994) consider a model with similar properties. Yet, they assume a peculiar type of preferences: when a buyer buys one of the units, say an apartment, even if this unit is not the preferred unit and has been obtained for free, this buyer will not try to buy the more preferred unit even if that could be done again at a extremely low price. This amounts to an extreme assumption of costly disposal. For a general analysis with no strategic behavior, see also Gul and Stacchetti (2000), where a negative result for existence of efficient, strategy-proof mechanisms with general preferences (or even restricting to no complementarities and no income effects) is provided.
Buyers who like an apartment facing west more than an apartment facing east would be willing to pay more for the former than for the latter. On the other hand, if buyers’ main goal is to buy a place to live in, once they own one apartment, say one facing east, they would still be willing to pay for an apartment facing west (the value of an ”upgrade”), but not as much as what they would if they did not own an apartment yet.

To the best of my knowledge, this issue has so far suffered from neglect. In this paper, first I study what allocation of objects to buyers would be efficient in this set up. That is, how the objects should be assigned to buyers in order to maximize the gains from trade. Then I show that this efficient allocation can be implemented with oral, ascending auctions for the right to choose, also known as ”pooled” auctions, which in fact are commonly used to sell condominium units (see Ashenfelter and Genesove (1992)). These are sequential, English auctions in which the winner of a round does not win a specific object, but the right to choose any of the yet unsold objects.

Efficient auctions, however, are not optimal from the seller’s point of view in this case, even if we disregard the possibility of leaving the object unsold and assume symmetry among buyers. Indeed, even if buyers are symmetric ex-ante, once their most preferred object has already been taken, bidders become weaker competitors vis-à-vis buyers whose most preferred objects are yet unsold. Thus, in line with what we know about asymmetric auctions and incentive problems in general, I will show that distorting the allocation to favor these bidders helps reducing the informational rents that other, stronger bidders obtain.

These ”optimal distortions” may have little practical appeal. As in the asym-
metric buyers case in auctions for a single object, they require complex auction rules, and a detailed knowledge of the distribution of buyers’ preferences. From a more practical point of view, however, there are simple designs that induce distortions in this direction and consequently increase the seller’s revenues. Indeed, I show that the seller can increase her revenues by auctioning all the rights to choose before any one is exerted, as is often done in real estate auctions.3

As an incidental result, this paper also explains the tendency of prices in successive rounds to decline, as observed for these same auctions by Ashenfelter and Genesove (1992). By now we are used to find that the once termed “declining price anomaly” is not so much of an anomaly. Yet, in this paper I show that in right-to-choose auctions rather than anomaly declining prices should be considered as the rule. After all, the value of what is offered for sale, the right to choose among the shrinking set of objects not sold before, declines too.

Several assumptions underlie these results. The most restrictive of all is the unidimensionality of preferences. That is, I assume throughout that if two different buyers rank the objects in the same way and are willing to pay the same for their most preferred one, then they are also willing to pay the same for the less preferred one. Absent this perfect correlation, our efficiency result would not hold, and would have to be considered only as an approximate result. Whether other complex auction formats can attain efficiency for general preferences is an open question. Yet, from the analysis of Gul and Stacchetti (2000) one can suspect that the prospect is indeed dim.

In the next section I present the model and main assumptions. Section 3 studies efficiency and shows that right-to-choose auctions implement the efficient allocation. Section 4 considers the problem from the perspective of the seller’s revenue. Then Section 5 discusses price patterns and the comparison with homogeneous goods. Finally, Section 6 concludes with a discussion of extensions.

3Efficiency requires that winners of early rounds make their choices before the beginning of later rounds.


2 The model

There are $n$ potential buyers for two distinct objects, $I = E, W$ (East and West). Buyer $i$’s preferences, $i = 1, 2, .., n$ are characterized by two parameters $(v_i, \theta_i)$, where $\theta_i \in \{E, W\}$, is the “location type”, and $v_i \in [0, 1]$, is the “valuation type”. For each $i$, $\theta_i$ takes the value $E$ with a probability of $\frac{1}{2}$ and the value $W$ with a probability of $\frac{1}{2}$. Also, $v_i$ is the realization of a random variable with c.d.f. $F$ and density $f$ which takes positive values on $[0, 1]$. All random variables are independent. All this is common knowledge. In addition, buyer $i$ knows the realization of $(\theta_i, v_i)$.

The location type of a buyer determines her most preferred object. The valuation type determines how much the buyer is willing to pay for that object. We also assume away income effects in the preferences of buyers. Thus, the pay-off for buyer $i$ with parameters $(\theta_i, v_i)$ who obtains object $I = \theta_i$, or both objects at a price $P$ is $v_i - P$. The parameter $v_i$ also determines $i$’s willingness to pay for the less preferred object, $l(v_i)$. Thus, the pay-off for this buyer if she obtains her less preferred object $I \neq \theta_i$ (and only this object) is $l(v_i) - P$, where $l$ is a strictly monotone increasing function. We assume $l(0) = 0$. Also, $1 > l'(v)$. For instance, the willingness to pay for the less preferred object may be a fixed proportion of the willingness to pay for the most preferred.\footnote{Notice that, as opposed to what Gale and Hausch (1994) (implicitly) assume, a buyer is willing to pay a positive amount to get her most preferred object even if she already owns her less preferred one.} \footnote{In Section 6, I discuss the extension of this "unit demand" model for a case where the buyer is willing to pay a positive amount for the less preferred object even when she already owns her most preferred one.} Finally, the pay-off for buyer $i$ who pays $P$ and gets no object is $-P$.

We are portraying a situation in which, although the two objects can be identified (they are distinct, as two apartments in the same condominium would be), they are symmetric from an ex-ante point of view. That is, neither of them

is ex-ante more valuable than the other. Also, notice that buyers are assumed risk neutral and ex-ante symmetric. Finally, for simplicity we assume that the seller’s valuation for both objects is sufficiently small (non-positive). We next turn to discussing efficiency and its implementation in this setting.

3 Efficiency: oral, ascending auctions for the right-to-choose

Efficiency under these assumptions is easy to characterize. Without loss of generality, assume throughout that buyer 1 has the highest valuation type, buyer 2 has the second highest, etc. Then, if $\theta_1 \neq \theta_2$ (buyers 1 and 2 prefer different objects), buyers 1 and 2 should receive their most preferred objects. If $\theta_1 = \theta_2$ then we may have to consider a third buyer: the one with highest valuation type among those with location type other than $\theta_1$. Call this buyer buyer $d_i$ (for different). Then again buyer 1 should get her most preferred object, but who gets the other depends now on which of the values: $l(v_2)$, or $v_{d_i}$ is highest. It should go to buyer 2 in the first case, and to buyer $d_i$ in the second.

Consider now selling the objects using an oral, ascending auction for the right to choose. As usual, we model English auctions by assuming that a clock points to continuously ascending prices, and buyers keep their hands up for as long as they wish. The round ends once all but one of them lower their hands. The winner is this remaining buyer. The price is the one shown in the clock at that moment.

The oral ascending auction for the right to choose is a common mechanism used in sales of condominium units. It consists of a sequence of regular English auctions in which the winner does not get any prespecified object, but the right to choose any one among the yet unsold objects. In our case, we would have only two rounds. It is not difficult to see that the winner of the first would certainly
choose her most preferred object. Also, if every bidder observes the choice of this winner, the best strategy for every buyer in the second (last) auction is to bid up to her willingness to pay for the remaining object. Thus, if the buyers’ equilibrium strategy in the first auction is such that the winner is the buyer with highest $v_i$, independently of $\theta_i$, then this mechanism indeed implements an efficient allocation. The first proposition simply states that this is the case.

**Proposition 1** When choices are made before new rounds begin, the oral, ascending auction for the right to choose has an efficient equilibrium.

**Proof.** See Appendix. ■

The equilibrium strategies are cumbersome to write but simple to interpret. As usual, the dropping price $P$ should satisfy the property that the buyer is indifferent between dropping at that price and remaining in the auction for a little longer. The two alternatives result in different outcomes for the buyer only if all remaining buyers are considering dropping at (approximately) that same price. In a monotone, continuous equilibrium, this happens when all remaining buyers have the same valuation type (but perhaps not the same location type) as the buyer considered. On the other hand, again under monotonicity, the dropping prices of buyers who have already dropped reveal their valuation type. Then, these are the conjectures under which the equilibrium dropping price is computed by all buyers. The only novelty in our set-up is that valuation types $v_i$ do not determine the other type dimension: location type $\theta_i$. So what happens in the second period is not completely deterministic even after conditioning on the observations and conjectures made in the first round. Some randomness remains.

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6Strategies are described by vectors of functions, one for each number of bidders that have already dropped. Each function gives the dropping price for the buyer under the assumption that no other drop occurs, and given the valuations of drop-outs inferred from their dropping price and the valuation type of the buyer. Monotonicity here means that all those functions are monotone. Continuity means also that when one more drop occurs the new dropping price for a buyer that was indifferent between dropping and staying is unchanged.

6
We are able to show that strategies computed in this way are monotone, continuous, and are indeed equilibrium strategies. Then the efficiency of right-to-choose auctions follows.

As an illustration, consider the three buyer case. Assume no buyer has yet dropped. Then, the dropping price $P_0(v_i)$ for a buyer $i$ with valuation $v_i$ (and any $\theta_i$) equates what the buyer gets by dropping now under the conjecture that the rest of the buyers have valuations $v_i,$

$$\left(\frac{1}{2}\right)^2 [v_i - l(v_i)],$$

with what the buyer expects by staying under that same assumption,

$$[v_i - P_0(v_i)].$$

Indeed, by dropping the buyer gets positive surplus only if the other buyers have a location type different than her own type. In this event, which has probability $\frac{1}{4},$ one of the other bidders (the loser) will bid $l(v_i)$ in the second round. In any other case, either the buyer gets no object in the second round or she has to pay a price equal to her own willingness to pay for the object left. On the other hand, if she decides to stay, the buyer gets to choose her most preferred object, since under the equilibrium conjectures she will win the right to choose first. Thus, the dropping price in the first round is

$$P_0(v_i) = \frac{3v_i + l(v_i)}{4}.$$ 

This is an increasing function of $v_i.$ Similarly, one can compute the dropping price $P_1(v_i; v_3)$ for a buyer with valuation type $v_i$ when one other bidder has already dropped at a price that reveals (according to the above equation) a valuation of $v_3.$

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7If either of the remaining two buyers has the same location type, either that buyer wins and takes the most preferred object of buyer $i,$ or that buyer will bid in the second period her valuation for the remaining object, which will be no lower than that of buyer $i$: they have the same location and (under equilibrium conjectures) valuation types.
Assume, for instance, that $v_3 > l(v_i) > l(v_3)$. That is, assume that when buyer 3 dropped the clock was approaching the dropping decision of buyer $i$. Then, again under the assumption that the remaining competitor also has valuation $v_i$, if buyer $i$ drops she will obtain the second object only if:

a) this remaining competitor has different location type in which case buyer $i$’s rent will be either $(v_i - l(v_3))$ or $(v_i - v_3)$, depending on the location type of buyer 3, each with equal probability; or

b) if the remaining competitor and buyer 3 have the same location type as buyer $i$, in which case buyer $i$ will obtain a surplus of $((l(v_i) - l(v_3))$.

This means expected surplus of

$$
\left(\frac{1}{2}\right) (v_i - \frac{1}{2}l(v_3) - \frac{1}{2}v_3) + \left(\frac{1}{2}\right)^2 (l(v_i) - l(v_3)).
$$

By staying longer under the same conjectures she is certain to get her most preferred object, and then surplus $(v_i - P)$. Thus,

$$
P_1(v_i; v_3) = \frac{2v_i + v_3 + 2l(v_3) - l(v_i)}{4}.
$$

Again, this is an increasing function of $v_i$. Moreover, when $v_i = v_3$, $P_1(v_i; v_3) = P_0(v_i)$. That is, when one buyer drops (buyer 3), a buyer with strictly higher valuation will still wait a little longer before dropping. This continuity guarantees that the two remaining bidders do not drop at the same time, and then guarantees that the buyer with the highest valuation wins the first round.

4 Maximizing the seller’s revenue

We now investigate how a seller should design her selling mechanism in order to maximize her expected revenue. This section generalizes well known results on revenue equivalence and the allocation properties of optimal auctions (see

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8If $l(v_i) > v_3$, we should add to this $\left(\frac{1}{2}\right) (l(v_i) - v_3)$, and change the expression for $P_1$ accordingly.
Myerson, 1981) to encompass the sale of substitutes. For any mechanism, denote by $p_i^I(v; \theta)$ the probability that buyer $i$ obtains object $I$ when location types are $\theta = (\theta_1, \theta_2, ..., \theta_n)$ and valuation types are $v = (v_1, v_2, ..., v_n)$. Also, denote by $\Omega_{-i}$ the type space for $n - 1$ buyers, $\Omega_{-i} = ([0, 1] \times \{E, W\})^{n-1}$, and by $\mu$ the probability measure induced on this set by the random variables $v_j, \theta_i, j \neq i$. Also, let

$$
\rho_i^E(v_i; \theta_i) = \int_{\Omega_{-i}} p_i^E((v; \theta)_{-i}; (v_i, \theta_i)) \left[ 1 - p_i^W((v; \theta)_{-i}; (v_i, \theta_i)) \right] d\mu(\Omega_{-i}),
$$

denote the expected probability that buyer $i$ obtains object $E$ only, given that her type is $(\theta_i, v_i)$. Define similarly $\rho_i^W(v_i; \theta_i)$. Finally, let

$$
\rho_i^B(v_i; \theta_i) = \int_{\Omega_{-i}} p_i^E((v; \theta)_{-i}; (v_i, \theta_i)) p_i^W((v; \theta)_{-i}; (v_i, \theta_i)) d\mu(\Omega_{-i})
$$

be the probability that buyer $i$ obtains both objects.

**Lemma:** (Revenue equivalence) Two mechanisms with common $\rho_i^I$, $I = E, W$, and $\rho_i^B$ are revenue and rent equivalent if each buyer $i$ with type $(0, \theta_i)$, $\theta_i = E, W$ obtains the same rents in both. Buyer $i$’s expected rents when $\theta_i = E$ are given by

$$
\pi_i(v_i; E) = \int_0^{v_i} [\rho_i^E(z; E) + \rho_i^B(z; E) + l'(z)\rho_i^W(z; E)] dz + \pi_i(0; E),
$$

and similarly when $\theta_i = W$.

**Proof.** See Appendix. ■

Consider now the problem of maximizing the seller’s expected revenue in the set of all possible auctions. As usual, notice that this is equivalent to maximizing the difference between total surplus (gains from trade) and expected buyers’ rents. Define the function $J : R \rightarrow R$, the ”virtual valuation” (see, for instance, Myerson, 1981), as

$$
J(v) = v - \frac{1 - F(v)}{f(v)},
$$

and define $J_l : R \rightarrow R$ as

$$
J_l(v) = l(v) - \frac{1 - F(v)}{f(v)}l'(v).
$$
We make the traditional regularity assumption, that $J(v_i)$ is increasing in $v_i$, but also that $J_l(v_i)$ is increasing in $v_i$\textsuperscript{9}. As we will see, here efficiency will not be synonym of optimality even under this assumption. However, with our monotonicity assumptions, we can characterize the type of distortions that maximize the seller’s revenue. We restrict attention to auctions that assign the objects to buyers with probability 1, so that we do not have to consider the reservation value of the seller or the commitment not to sell. The extension to include the possibility of not selling is straightforward. Thus, the following proposition offers the characterization of optimal auctions. We still use the convention that $v_1$ is the highest realization of the valuation type, $v_2$ the second largest, etc. Also, again let $d_i$ be the buyer with the highest valuation type among those with location type other than $\theta_1$.

**Proposition 2** An optimal auction assigns buyer 1 her most preferred object. If $\theta_1 \neq \theta_2$, then the optimal auction assigns the second object to buyer 2. If $\theta_1 = \theta_2$, then the optimal auction assigns the second object to buyer 2 or buyer $d_i$, respectively if $J_l(v_2)$, or $J(v_{d_i})$ (if $d_i$ exists) is higher.

**Proof.** See Appendix. ■

As opposed to what happens in the standard symmetric, independent private value case, this proposition means that optimal (from the seller’s viewpoint) auctions for substitutes will be inefficient. In particular, under our assumptions,

**Corollary 2:** The optimal auction allocates the two objects to buyers with the same location type more often than efficient.

Indeed, if $l(v_2) = v_{d_i}$, then $J_l(v_2) > J(v_{d_i})$. This distortion should sound familiar: in the second auction, a buyer with location type equal to $\theta_1$ is ”weaker” in that she obtains her willingness to pay (for the second object) from a less favorable distribution.

\textsuperscript{9}Monotonicity of the inverse hazard rate $\frac{1-F(v)}{f(v)}$ and concavity of $l$ already imply this.
For the sake of illustration, consider the case where \( v_i \) is distributed uniformly, \( l \) is linear: \( l(v_i) = lv_i \), and \( n = 3 \). In this case \( J(v_i) = 2v_i - 1 \) and \( J_l(v_i) = lJ(v_i) \). Consider the following modifications to the oral, ascending action for the right to choose:

- The last buyer to drop (who sets the price in the first round) has to pay a fee that depends on the dropping price of buyer 3,

\[
\phi(v_3) = \begin{cases} 
\frac{1-l}{2} & \text{if } v_3 \leq \frac{1}{2} \\
\frac{1-l}{2} - (v_3 - \frac{1}{2}) \frac{3(1-l)}{4} & \text{if } v_3 > \frac{1}{2}
\end{cases}
\]

in order to participate in the second round. (The fee is equal to the expected payoff if buyer 2 has valuation type \( v_3 \) too.)

- The first buyer to drop is handicapped by \( \frac{1-l}{2} \) in the second round. That is, she loses that round unless she beats the other buyer (buyer 2) by at least that amount.

Using the same notation of the previous section, then equilibrium behavior means that buyers drop at prices

\[
P_0(v_i) = \begin{cases} 
v_i & \text{if } v_i \leq 1/2 \\
lv_i + \frac{1-l}{2} & \text{if } v_i > 1/2
\end{cases}
\]

and

\[
P_1(v_i; v_3) = \begin{cases} 
\frac{v_i + v_3 - l(v_i - v_3)}{2} & \text{if } v_3 \leq 1/2 \\
\frac{v_i + v_3 - l(v_i - v_3)}{2} - \frac{3}{4} (1-l) \left( v_3 - \frac{1}{2} \right) & \text{if } v_3 > 1/2 \\
\frac{v_i + v_3 - l(v_i - v_3)}{2} - (1-l) \left( v_3 - \frac{1}{2} \right) & \text{if } v_3 > 1/2 \\
\text{and } lv_i + \frac{1-l}{2} \geq v_3 & \text{and } lv_i + \frac{1-l}{2} < v_3
\end{cases}
\]

Again, we can check both continuity and monotonicity of these two functions, so that \( v_3 \) is trivially inferred from \( P_0 \), and also \( P_0(v_i) = P_1(v_i; v_i) \), so that the first round is won by the highest valuation buyer. It is also straightforward to check that this is an optimal auction for this particular case.
As in the problem of designing optimal auctions for a single object when buyers are asymmetric (see, for instance, Maskin and Riley (1984)), the type of favoritism that buyers should enjoy depends on the types of the buyers (perhaps inferred from their bid behavior). Also, detailed information about the distribution of this types is needed. This together with the complexity of the optimal mechanism in general explains why we hardly observe auctions like the one outlined above.

However, an important, practical question is whether other simple auctions may improve upon the unmodified oral, ascending auction for the right to choose (i.e., upon an efficient auction).\textsuperscript{10} We show next that indeed there is a simple modification to our right to choose auction that attains this goal, one that in fact is commonly observed in auctions of real estate. This modification consists of simply auctioning all rights to choose before any one is exerted. That is, in our two object case, the choice of the winner in the first round would not yet be known when bidders bid in the second round. It is straightforward to show that the winner in round $n = 1, 2$ would be $i = n$. That is, buyer 2 would win the second object even if buyer 3 is willing to pay more for this object (i.e., when $\theta_1 = \theta_2 \neq \theta_3$, and $l(v_2) < v_3$). This type of inefficiency is indeed in line with the ones that an optimal mechanism would induce, and in fact,

**Proposition 3** In an auction for the right to choose, keeping secret buyer 1’s choice increases the expected revenues for the seller.

**Proof.** See Appendix. ■

To understand the way these distortions affect the prices, again let us revisit our $n = 3$ case. Apart from the value of choosing in first place ($\frac{v_i - l(v_2)}{2}$), the

\textsuperscript{10}Standard sequential auctions, where objects, and not the right to choose, are auctioned in sequence, are not good alternatives. Indeed, coordination problems should render this auction format inefficient (high valuation bidders may obtain both objects, if rival bidders have sufficiently low valuation, for instance), but in ways that do not help raising the seller’s revenue.
price is again determined by the cost for buyer 2 of losing the first right to choose. If the choice of buyer 1 is not known at the moment of bidding for the second object, then this cost is simply $\frac{v_3 + l(v_3)}{2}$, the valuation (i.e., bid) of buyer 3. However, if the choice is announced, this cost is again $l(v_3)$ with probability $\frac{1}{2}$ (when $\theta_1 = \theta_3$), $v_3$ with probability $\frac{1}{4}$ (when $\theta_1 \neq \theta_2, \theta_3$), and with probability $\frac{1}{4}$ (when $\theta_2 = \theta_1 \neq \theta_3$) the cost is $\min\{v_3, l(v_2)\}$ (either buyer 2 loses also the second good, when $l(v_2) < v_3$, or she has to pay $v_3$, in the opposite case). Thus, the expected cost is now only $\frac{v_3 + l(v_3) + \min\{v_3, l(v_2)\}}{2}$. This is lower than the expected cost of losing in the inefficient auction, and the difference (times two, since there are two units for sale) is the increase in the seller’s revenues for this case.

5 Discussion

5.1 Declining prices

A tendency for the sequence of prices of repeated sales to decline has been widely observed. In particular, Ashenfelter and Genesove (1992) report this pattern of prices in sales of condominium units when sellers use right-to-choose auctions. The authors are quick to point out that such price behavior should be expected if the units are not homogeneous, so that early winners can buy superior units.\textsuperscript{11} However heterogeneity of the units does not entirely explain the downward drift, according to the authors. Here, we have been considering a case of “symmetric” (from an ex-ante point of view, of course) objects. Let us return to our three buyer case with observed choices of previous winners (the exercise without observability is completely analogous). As we have discuss in the previous subsection, the price in the second round coincides with the opportunity cost for buyer 2 of losing the

\textsuperscript{11} With multi-unit demands, prices should also be expected to decline. See, for instance, Bernhardt and Scoones (1994), or Burguet and Sákovics (1997). Black and de Meza (1992) obtain this same result when the seller offers the "buyer’s option".
first round, net of the option value of choosing first. Thus, buyer 2’s bid in the
first round (and then the price in that round) is that same opportunity cost but
this time without deducting the option value of choosing first (something that
buyer 2 can only obtain in this first round). Thus, again the price in the first
round is \( \frac{v_2 - l(v_2)}{2} \) higher than the price in the second round. Even though the two
objects are stochastically equivalent in terms of value from an ex-ante point of
view, the ability to choose has a price. When both objects are indistinguishable
\((l(v) = v)\), this difference is nil, and we recover the martingale property of prices
in sequential auctions.

5.2 The value of taste diversity

When \( l(v) = v \), the model we have been considering reduces to the standard two
unit auction with unit demand bidders. Taking this as a benchmark, the case
where \( l(v) < v \) can be seen as a reduction of value of the objects for sale. Indeed,
the buyers’ willingness to pay for the objects, on average, is lower. It is natural
to expect that the maximum revenue that a seller can extract in this second case
is lower. In fact, they need not be.

If buyers are indifferent among the objects, then the expected revenue of the
revenue maximizing auction is simply twice the expected value of the third order
statistic. That is, the expected value of \( 2v_3 \). Indeed, under the assumption that
the objects are sold with probability 1,\(^{12}\) any efficient mechanism is optimal and
extracts that revenue. However, if buyers value one of the objects at \( l(v) \), as in
our case, our right to choose auction without observability of choices (which is
not even optimal) extracts revenue equal to \( v_3 + l(v_3) + \frac{v_2 - l(v_2)}{2} \). This is \( \frac{v_2 - l(v_2)}{2} - (v_3 - l(v_3)) \) in excess of \( 2v_3 \). The intuition for this difference is simple. Once
again, the third bidder sets the value of the two objects in the standard case, but

\(^{12}\)Again, this would also be optimal without this constraint provided that the reservation
value of the seller is low enough. The extension to allow for not selling is also straightforward.
when $l(v) < v$, the third bidder valuation for “some” object is on average $\frac{v_3 + l(v_3)}{2}$, instead of $v_3$. This accounts for the second term in the difference. However, when $l(v) < v$, the value is not set by bidder 3 completely. Indeed, as we have already mentioned, the price includes bidder 2’s willingness to pay for the right to choose in first place, $\frac{v_2 - l(v_2)}{2}$, and this is the first term of the difference. This difference may well be positive. For instance, if $n = 3$, $v$’s are uniform, and $l(v) = v(1 - \frac{v}{10})$, this difference is .005.

6 Concluding remarks

We have investigated the implementation of efficient and optimal allocations for the condominium problem. That is, for the sale of substitutes. Right to choose auctions, which are commonly used for this type of sales, are efficient if the winner of a round announces her choice prior to the sale of the next right to choose. However, keeping these choices secret until all the rights have been auctioned, as is usually done in practice, increases the seller’s revenue. In line with the type of distortions induced by the optimal mechanism, characterized in Section 4, keeping choices secret assign the goods to buyers with the same preferred objects more often than what is efficient.

We have assumed throughout that buyers have unit demands. Indeed, buyers were willing to pay a positive amount to “upgrade” their unit. That is, to buy the most preferred object even when they already owned the other object. But buyers were indifferent between owning two objects and owning their most preferred one. Most of the results above would extend easily to a scenario where buyers are also willing to pay a positive amount for the less preferred object even when they already own their favorite. Assume that a buyer with valuation $v$ is willing to pay $b(v) > v$, with $l' > l' - 1 > 0$. One can easily show that the auction for the right to choose where choices are announced before subsequent rounds
also implements the efficient allocation. This allocation would require comparing \( b(v_1) \) with \( v_2 \), or \( l(v_2) \) and \( v_{di} \) (depending on whether \( \theta_2 \) equals \( \theta_1 \) or not) to decide who obtains the second object.

Revenue equivalence also holds under this assumption. The optimal mechanism for this case adds a new type of inefficiency: excessive bundling. That is, buyer 1 obtains the two objects even when buyer 2 is willing to pay more for the second object. With multiple unit demands, keeping choices secret, however does not unambiguously increase the seller’s revenues.

We have also assumed that signals were unidimensional, which is certainly a strong assumption. Indeed, if preferences were multidimensional (if buyers with the same willingness to pay for their favorite object had different willingness to pay for the other) the right to choose auction could not possibly be efficient. In fact, no simple mechanism would. Thus, the results in this paper should be understood as approximate, but not exact, when the willingness to pay for the two objects are correlated, but not perfectly correlated.

References


Appendix

Proof of Proposition 1: The second round is trivial from the strategic point of view: buyer’s dominant strategy is to bid up to willingness to pay for whichever object is left. For the first round, we define the equilibrium strategies inductively. Assume $j < n - 2$ buyers have dropped revealing valuation types $(v_n, v_{n-1}, ..., v_{n-j+1})$. This includes trivially the case that no buyer has dropped yet. We will see that indeed valuation types can be inferred from dropping prices. Let $k, n \geq k \geq n - j$ be such that $v_k \geq l(v_i) \geq v_k + 1$. That is, $k$ is the highest index of the buyer among the ones that have already dropped and whose willingness to pay for her most preferred object exceeds buyer $i$’s willingness to pay for her less preferred object. When $l(v_i)$ is higher than $v_m$ for $m = n, n-1, ..., n-j+1$, then $k = n - j$.

We compute the dropping price for a buyer with valuation type $v_i$ similarly to the way we did for the case $n = 3$. That is, we assume that the buyer conjectures that all other buyers that remain in the auction have valuation $v_i$ too. Under this conjecture we compute how much the buyer expects if dropping at the current price and how much she gets by staying. By staying the buyer wins the first round at the going price. So we also have to check that indeed under the conjectures of the buyer, all other buyers’ equilibrium behavior is to drop at the current price. Thus:

- Staying: the price will be the current price $P$, the buyer chooses her most preferred object. Thus, staying means a pay-off of $v_i - P$.

- Dropping: one buyer will get her most preferred object, which may be buyer $i$’s most preferred object or not, each event with probability $1/2$. Then other $n - j - 2$ buyers have valuation type $v_i$. When $j < n - 2$, buyer $i$ will not get a positive payoff (either because the price in the second round will reach her willingness to pay or because she will be defeated by other bidders) unless the winner and the other $n - j - 2$ buyers with valuation $v_i$ have different location type from that of buyer $i$. In this latter case, buyer $i$ wins the second (her most
preferred) object, and the price will go up to \( l(v_i) \), \( v_k, v_{k-1}, \ldots \) or \( v_{n-j+1} \), depending on whether all buyers \( k, \ldots, n-j+1 \) have also the same location type of the winner, or all \( k-1, \ldots, n-j+1 \) have that location type but not buyer \( k \), etc. Thus, for \( j < n-2 \), the pay-off expected if dropping is

\[
\left( \frac{1}{2} \right)^{n-j-1} \left\{ \sum_{m=1}^{k-(n-j)} \left( \frac{1}{2} \right)^m [v_i - v_{n-j+m}] + \left( \frac{1}{2} \right)^{k-(n-j)} [v_i - l(v_i)] \right\} = \\
\left( \frac{1}{2} \right)^{n-j-1} v_i - \left\{ \sum_{m=n-j+1}^{k} \left( \frac{1}{2} \right)^{m-1} v_m + \left( \frac{1}{2} \right)^{k-1} l(v_i) \right\}.
\]

Thus, for \( j < n-2 \), we propose

\[
P_j(v_i; v_n, v_{n-1}, \ldots, v_{n-j+1}) = \\
v_i \left( 1 - \left( \frac{1}{2} \right)^{n-j-1} \right) + \left\{ \sum_{m=n-j+1}^{k} \left( \frac{1}{2} \right)^{m-1} v_m + \left( \frac{1}{2} \right)^{k-1} l(v_i) \right\}.
\]

When \( j = n-2 \), that is, when only one other bidder is left, things are slightly different. In particular, buyer \( i \) could obtain a positive pay-off even when her location type coincides with that of buyer 1. Thus, let \( k' (\geq k) \) be the integer that satisfies \( v_{k'} \geq l(v_3) \geq v_{k'} + 1 \). The pay-off when dropping in this case is

\[
\frac{1}{2} v_i + \left( \frac{1}{2} \right)^{k-1} l(v_i) - \left\{ \sum_{m=3}^{k} \left( \frac{1}{2} \right)^{m-1} v_m + \sum_{m=k+1}^{k'} \left( \frac{1}{2} \right)^{m-2} v_m + \left( \frac{1}{2} \right)^{k'-2} l(v_3) \right\},
\]

and then,

\[
P_2(v_i; v_n, v_{n-1}, \ldots, v_3) = \\
\frac{1}{2} v_i - \left( \frac{1}{2} \right)^{k-1} l(v_i) + \left\{ \sum_{m=3}^{k} \left( \frac{1}{2} \right)^{m-1} v_m + \sum_{m=k+1}^{k'} \left( \frac{1}{2} \right)^{m-2} v_m + \left( \frac{1}{2} \right)^{k'-2} l(v_3) \right\}.
\]

All the functions defined above are monotone in \( v_i \). Also \( P_j(v_i; v_n, v_{n-1}, \ldots, v_{n-j+1}) = P_{j+1}(v_i; v_n, v_{n-1}, \ldots, v_{n-j+1}, v_{n-j} = v_i) \). That is, if a drop \( j + 1 \) happens exactly when buyer \( i \) was planning to drop, buyer \( i \), but not any buyer with valuation higher than hers, will also drop. These were the two conjectures that we had to check. From this too, optimality of behavior follows trivially: dropping before
the corresponding price only makes a difference when the expected pay-off in the first round is higher, and dropping after that price only makes a difference when the expected pay-off in the second round is higher. QED.

**Proof of Lemma** (Revenue equivalence):

The proof is very standard: Let $X_i(v_i; \theta_i)$ denote the expected payments of buyer $i$ with valuation type $v_i$ and location type $\theta_i$. For a location type $E$, for instance, incentive compatibility requires (among other things) that

$$v_i = \arg \max_z \left[ \rho_i^E(z; E) + \rho_i^B(z; E) \right] v_i + \rho_i^W(z; E) l(v_i) - X_i(z; E).$$

Thus

$$\frac{\partial X_i(v_i; E)}{\partial v_i} = \frac{\partial \left[ \rho_i^E(v_i; E) + \rho_i^B(v_i; E) \right]}{\partial v_i} v_i + \frac{\partial \rho_i^W(v_i; E)}{\partial v_i} l(v_i),$$

which, integrating by parts, implies

$$\pi_i(v_i; E) = \int_0^{v_i} \left[ \rho_i^E(z; E) + \rho_i^B(z; E) + l'(z) \rho_i^W(z; E) \right] dz + \pi_i(0; E).$$

The Lemma follows. QED.

**Proof of Proposition 2:**

We can write the expected revenue for the seller as the difference between surplus and expected rents for the buyers. Thus, the sellers revenue are given by

$$\sum_{i=1}^n \sum_{\theta_i = E, W} \frac{1}{2} \left\{ \int_0^{v_i} \left[ \rho_i^\theta_i(v_i; \theta_i) + \rho_i^B(v_i; \theta_i) \right] v_i + \rho_i^W(v_i; \theta_i) l(v_i) \right\} f(v_i) dv_i - \pi_i(0; \theta_i) \right\},$$

where $\theta_i$ represents the location type opposite to $\theta_i$. Inverting the order of integration in the double integral, the above expression can be written as

$$\sum_{i=1}^n \sum_{\theta_i = E, W} \frac{1}{2} \left\{ \int_0^{v_i} \left[ \rho_i^\theta_i(v_i; \theta_i) + \rho_i^B(v_i; \theta_i) \right] J_i(v_i) + \rho_i^W(v_i; \theta_i) J_i(v_i) f(v_i) dv_i - \pi_i(0; \theta_i) \right\}.$$
Proof of Proposition 3:

First we compute equilibrium strategies for the right to choose auction when the choice of the winner in the first round is not revealed. We conjecture that the bidding behavior of the first round reveals no information about the location type of the winner. In the second round, bidders bid their willingness to pay for an object the is of type $E$ with probability $1/2$ and of type $W$ with probability $1/2$. That is, a bidder with valuation type $v$ bids $\frac{v + l(v)}{2}$. In the first round, we compute the equilibrium bidding behavior using the same inferences as we used in the proof of Proposition 1. Thus, if no more than $n - 2$ other bidders have dropped (i.e., if there are still 2 or more other bidders left), a bidder expects no profit in the second round of the auction. Thus, she stays up to $P = v$. If only 1 other bidder is left in the auction, a bidder computes what she expects staying and what she expects dropping under the conjecture that the other bidder still active has her same valuation $v$. If the last bidder to drop so far has done so at a price $v_3$, dropping means expected rents of $\frac{v + l(v) - (v_3 + l(v_3))}{2}$. As before, staying means expected rents of $v - P$. Thus, the dropping price in the first round would be

$$P_1(v; v_n, v_{n-1}, ..., v_3) = \frac{v_i - l(v_i) + (v_3 + l(v_3))}{2}.$$

Again, this is a strictly increasing function with $P_1(v_3; v_n, v_{n-1}, ..., v_3) = v_3$, independent of the location type of the bidder. Thus, we have obtained indeed an equilibrium. In this equilibrium, buyer 1 wins the right to choose in first place, and buyer 2 obtains the object left. The price in the first round is $\frac{v_2 - l(v_2) + (v_3 + l(v_3))}{2}$, and the price in the second period is $\frac{v_3 + l(v_3)}{2}$.

In the efficient right to choose auction, the expected price in the first period is given by $P_2(v_2; v_n, ..., v_3)$ as defined in the proof of Proposition 1, and the expected price in period 2 is

$$(k - 2) \left( \frac{1}{2} \right)^{k-1} l(v_2) + \left( \frac{1}{2} \right)^{k'-2} l(v_3) + (m - 2) \sum_{m=3}^{k} \left( \frac{1}{2} \right)^{m-1} v_m + \sum_{m=k+1}^{k'} \left( \frac{1}{2} \right)^{m-2} v_m.$$
Thus, computing the difference between expected prices in the right to choose without observability and the expected price in the efficient auction, we obtain (for $n > 3$),
\[
\left(\frac{1}{2}\right) \max\{v_3 - l(v_2)\} + \left(\frac{1}{2}\right)^3 \max\{v_4 - l(v_2)\} - \sum_{m=6}^{n} (m-5) \left(\frac{1}{2}\right)^{m-1} \max\{v_m - l(v_2)\} - \sum_{m=4}^{n} \left(\frac{1}{2}\right)^{m-2} \max\{v_m - l(v_3)\},
\]
which is positive since $\max\{v_3 - l(v_2)\} > \max\{v_m - l(v_3)\}$ for $m > 3$, and $\max\{v_4 - l(v_2)\} > \max\{v_m - l(v_2)\}$ for $m > 4$, and $\sum_{m=6}^{n} (m-5) \left(\frac{1}{2}\right)^{m-1} < \left(\frac{1}{2}\right)^3$, whereas $\sum_{m=4}^{n} \left(\frac{1}{2}\right)^{m-2} < \left(\frac{1}{2}\right)$. QED.