Non-Revelation Mechanisms for Many-to-Many Matching: Equilibria versus Stability

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Abstract

We study many-to-many matching markets in which agents from a set $A$ are matched to agents from a disjoint set $B$ through a two-stage non-revelation mechanism. In the first stage, $A$-agents, who are endowed with a quota that describes the maximal number of agents they can be matched to, simultaneously make proposals to the $B$-agents. In the second stage, $B$-agents sequentially, and respecting the quota, choose and match to available $A$-proposers.

We study the subgame perfect Nash equilibria of the induced game. We prove that stable matchings are equilibrium outcomes if all $A$-agents’ preferences are substitutable. We also show that the implementation of the set of stable matchings is closely related to the quotas of the $A$-agents. In particular, implementation holds when $A$-agents’ preferences are substitutable and their quotas are non-binding.

Keywords: matching, mechanisms, stability, substitutability

JEL–Numbers: C78, D78.
1 Introduction

We study many-to-many matching markets in which agents from a set $A$ are matched to agents from a disjoint set $B$ through a two-stage non-revelation mechanism. In the first stage, $A$-agents, who are endowed with a quota that describes the maximal number of agents they can be matched to, simultaneously make proposals to the $B$-agents. In the second stage, $B$-agents sequentially, and respecting the quota, choose and match to available $A$-proposers. Mechanisms where the agents on one side of the market apply simultaneously and then the agents on the other side choose sequentially are very common, e.g., in college admission and school choice (Roth and Sotomayor, 1990; Vulkan et al., 2013).

We study the subgame perfect Nash equilibria of the induced game. We prove that stable matchings are equilibrium outcomes if all $A$-agents’ preferences are substitutable (Theorem 1); even if only one $A$-agent does not have substitutable preferences it can happen that some stable matching is not an equilibrium outcome (Example 1). We also show that the implementation of the set of stable matchings is closely related to the quotas of the $A$-agents (Theorem 2). In particular, implementation holds when $A$-agents’ preferences are substitutable and their quotas are non-binding (Corollary 1).

In the context of many-to-one matching between students and colleges, Romero-Medina and Triossi (2014) introduce two sequential non-revelation mechanisms. They show that if colleges’ preferences are substitutable, then the mechanisms implement the set of stable matchings in subgame perfect Nash equilibrium. More specifically, Romero-Medina and Triossi (2014) propose a mechanism, called the CSM (students apply Colleges Sequentially choose Mechanism), which coincides with our mechanism by taking the set $A$ to be students, the set $B$ to be colleges, and setting the quota for each agent (student) in $A$ to be equal to one. Assuming furthermore that preferences of the agents in set $B$ (colleges) are substitutable, Romero-Medina and Triossi (2014, Proposition 1) show that CSM implements the set of stable matchings. We provide examples that show that Proposition 1 of Romero-Medina and Triossi (2014) is tight in the sense that under a slight relaxation of the assumptions, implementation needs no longer be possible (Examples 2 and 3).

Romero-Medina and Triossi (2014) also consider a mechanism, called the SSM (colleges apply Students Sequentially choose Mechanism), where colleges first simultaneously propose to students and then students sequentially pick a college. The SSM coincides with our mechanism by taking the set $A$ to be colleges, the set $B$ to be students, and not limiting the quota for each agent (college) in $A$. Romero-Medina and Triossi (2014, Proposition 2) show that SSM implements the set of stable matchings. Our Corollary 1 generalizes Romero-Medina and Triossi (2014, Proposition 2).

Finally, in Section 4, we discuss the validity of our results when using the stronger stability notion of setwise stability instead of (pairwise) stability: while Theorem 1 remains valid, Theorem 2 does not hold anymore.
2 Preliminaries

2.1 Many-to-many matching

There are two disjoint and finite sets of agents $A$ and $B$. Let $I = A \cup B$ denote the set of agents. Generic elements of $A$, $B$, and $I$ are denoted by $a$, $b$, and $i$, respectively. The set of (possible) partners of agent $i$ is $T_i \equiv B$ if $i \in A$, and $T_i \equiv A$ if $i \in B$. The preferences of agent $i$ are given by a linear order $P_i$ over all subsets of set $T_i$, $2^{T_i}$.

Let $P_i$ denote the collection of all possible preferences for agent $i$. Since we fix the set of agents, a (many-to-many matching) market is given by a preference profile, i.e., a tuple $P = \{P_i\}_{i \in I}$. For each agent $i \in I$, let $R_i$ denote the ‘at least as desirable as’ relation associated with $P_i$, i.e., for each pair $j, k \in T_i$, $j R_i k$ if and only if $j = k$ or $j P_i k$. For each agent $i$ with preferences $P_i$, let $Ch(\cdot, P_i)$ be the induced choice function on $2^{T_i}$. In other words, for each set $T \subseteq T_i$, $Ch(T, P_i)$ is agent $i$’s most preferred subset of $T$ according to $P_i$. A set of agents $T \subseteq T_i$ is acceptable to agent $i$ at $P$ if $T R_i \emptyset$.

A matching is a mapping from the set of agents $I$ into $2^A \cup 2^B$ such that for each agent $a \in A$ and each agent $b \in B$, $\mu(a) \in 2^B$, $\mu(b) \in 2^A$, and $[a \in \mu(b) \iff b \in \mu(a)]$. For any agent $i \in I$, set $\mu(i)$ is called agent $i$’s match (at $\mu$). Next, we introduce (pairwise) stability. Since the matching markets we consider are based on voluntary participation, we require a matching to be individually rational. Formally, a matching $\mu$ is individually rational if for all agents $i \in I$, $Ch(\mu(i), P_i) = \mu(i)$. Matching $\mu$ is blocked by a pair (of agents) $(a, b) \in A \times B$, $a \not\in \mu(b)$, if for all agents $i, j \in \{a, b\}$ with $i \neq j$, $j \in Ch(\mu(i) \cup \{j\}, P_i)$. A matching $\mu$ is (pairwise) stable if it is individually rational and not blocked by any pair $(a, b) \in A \times B$. Let $\Sigma(P)$ denote the set of stable matchings. Note that the set of stable matchings $\Sigma(P)$ can be empty (see, e.g., Roth and Sotomayor, 1990, Example 2.7). A well-known sufficient condition for the non-emptiness of $\Sigma(P)$ is substitutability of all agents’ preferences. The preferences $P_i$ of an agent $i \in I$ are substitutable if for all sets $T' \subseteq T_i$ and for all agents $j, j' \in T'$ with $j \neq j'$, $[j \in Ch(T', P_i) \implies j \in Ch(T' \setminus \{j'\}, P_i)]$. For a subset of agents $I' \subseteq I$, we say that $P_{I'} \equiv (P_i)_{i \in I'}$ is substitutable if for all $i \in I'$, $P_i$ is substitutable.

2.2 A class of non-revelation mechanisms

We assume that for each agent $a \in A$, there is an exogenous quota, given by a positive integer $q_a$, so that any match for agent $a$ cannot have cardinality larger than $q_a$ (for instance due to legal or physical constraints). We suppose that $q_a$ is not smaller than the

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1 In other words, $P_i$ is transitive, antisymmetric (strict), and total.

2 In Section 4, we explain how our results would be affected if we used a stronger stability notion that is also often considered for many-to-many matching markets, setwise stability, instead of pairwise stability.

3 When formulating blocking like this we need to make sure $a$ and $b$ are not already matched (otherwise a matched pair could block).

4 Substitutability is an adaptation of the gross substitutability property (Kelso and Crawford, 1982) by Roth (1984) and Roth and Sotomayor (1990) to matching problems without monetary transfers.
largest acceptable match for agent $a$.\footnote{It could very well be that an agent might find matches that exceed a legally prescribed quota acceptable. We assume that, for all practical purposes, such an agent derives and uses “legal preferences” and a “legal choice function.”}

Let the set of agents $B = \{b_1, \ldots, b_k\}$. Let $\beta = (b_1, \ldots, b_k)$ be an order of the $B$-agents.

The \textit{[A simultaneously apply – B sequentially choose] mechanism $\varphi \equiv \varphi^{\beta,q}$}:

For each $a \in A$, let $r_a \equiv q_a$.

**Step 0 (applications):** $A$-agents simultaneously apply to sets of $B$-agents.

For each $a \in A$, agent $a$’s strategy is the set $s_a \in 2^B$ of $B$-agents agent $a$ applies to. Let $s_A = (s_a)_{a \in A}$.

**Steps $l = 1, \ldots, k$ (choices):** The set of $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents are the $A$-agents that applied to $b_l$ in Step 0 and that are still available, i.e., the set of agents $a \in A$ with $b_l \in s_a$ and $r_a > 0$. Agent $b_l$ chooses a subset of $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents. If an agent $a \in A$ is chosen by $b_l$, then they are (permanently) matched and we set $r_a \equiv r_a - 1$.

For each agent $b_l \in B$, agent $b_l$’s strategy is the choice function $s_b$ that for each $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$ describes agent $b_l$’s choice from the $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents.

For any strategy profile $s = (s_i)_{i \in I}$, the outcome of non-revelation mechanism $\varphi^{\beta,q}$ is a well-defined matching and the mechanism induces an extensive form game. Let $E^{\beta,q}(P)$ (or $E(P)$ if no confusion is possible) denote the set of subgame perfect Nash equilibria (SPE) at $P$, i.e., $E^{\beta,q}(P)$ is the set of subgame perfect Nash equilibria strategy profiles. Similarly, let $O^{\beta,q}(P)$ (or $O(P)$ if no confusion is possible) denote the set of SPE outcomes at $P$, i.e., $O^{\beta,q}(P)$ is the set of matchings that result from the set of SPE.

For any strategy profile $s$ and any agent $i \in I$, let $s_{-i} \equiv (s_j)_{j \in I \setminus \{i\}}$.

An example of a mechanism $\varphi^{\beta,q}$ is the application of students to public schools: a student cannot consume more than one school admission, but he is allowed to apply to more than one public school. Public schools process applications in sequence and once a student accepts an admission he is no longer available for later admissions.

### 3 Results

Our first result shows that when $A$-agents have substitutable preferences, the \textit{[A simultaneously apply – B sequentially choose] mechanism $\varphi^{\beta,q}$ implements in SPE a superset of the set of stable matchings.}

\footnote{Let $q, q'$ be two quota vectors. Then, $q \geq q'$ if and only if for all $a \in A$, $q_a \geq q'_a$.}
Theorem 1. (All stable matchings can be obtained as SPE outcomes)

For any $(\beta, q)$ and any preference profile $P$ where $P_A$ is substitutable,

$$\Sigma(P) \subseteq \mathcal{O}^{\beta,q}(P).$$

Examples 2 and 3 show that under the assumptions of Theorem 1, $\Sigma(P) \not\subseteq \mathcal{O}^{\beta,q}(P)$ is possible.

Proof. Without loss of generality, let $\beta = (b_1, \ldots, b_k)$. Let $P$ be a preference profile. Let matching $\mu$ be stable, i.e., $\mu \in \Sigma(P)$. Consider the following strategy profile $s$. Each agent $a \in A$ (only) applies to set $s_a \equiv \mu(a)$. For each $b_l \in B$ and for each of its decision nodes, let agent $b_l$ accept the set of $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents that he prefers most according to his preferences $P_{b_l}$. In view of the optimality of the decisions of the $B$-agents, it suffices to show that no agent $a \in A$ has a profitable unilateral deviation, i.e., he cannot get matched to a more preferred set of $B$-agents.

Suppose to the contrary that for some agent $a \in A$ such a deviation does exist. We show that then there exists a blocking pair for matching $\mu$. Let strategy $s'_a$ be the best possible deviation for agent $a$. Let strategy profile $s' = (s'_a, s_{-a})$ and matching $\mu' = \varphi^{\beta,q}(s')$. Since strategy $s'_a$ is a beneficial deviation, $\text{Ch}(\mu(a) \cup \mu'(a), P_a) \not\subseteq \mu(a)$. Let $b \in \text{Ch}(\mu(a) \cup \mu'(a), P_a) \setminus \mu(a)$. Note that $b \not\in \mu(a)$ and $b \in \mu'(a)$.

After agent $a$’s deviation, agent $b$ receives an application from $a$ and the set of previous applications (which by construction of strategy profile $s$ equals set $\mu(b)$). Then, in view of the optimality of agent $b$’s decision at strategy profile $s'$, it follows that $\text{Ch}(\mu(b) \cup \{a\}, P_b) = \mu'(b)$. Hence,

$$a \in \text{Ch}(\mu(b) \cup \{a\}, P_b). \quad (1)$$

For agent $a$, by substitutability of preferences $P_a$, $b \in \text{Ch}(\mu(a) \cup \mu'(a), P_a)$ implies

$$b \in \text{Ch}(\mu(a) \cup \{b\}, P_a). \quad (2)$$

Hence, (1) and (2) imply that $(a, b)$ is a blocking pair for $\mu$; a contradiction. \qed

The following example shows that substitutability of $P_A$ cannot be omitted in Theorem 1. In fact, even if only one $A$-agent does not have substitutable preferences, then it can happen that some stable matching is not an equilibrium outcome.

Example 1. ($P_A$ not substitutable and $\Sigma(P) \not\subseteq \mathcal{O}^{\beta,q}(P)$)

Consider the market with $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and preference profile $P$ given by Table 1: in this and the following examples, we list only individually rational matches and better matches are ranked higher. Note that all preferences except for those of agent $a_1$ are substitutable.
Table 1: Preference profile $P$ in Example 1

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
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<td>${b_2}$</td>
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<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${a_2}$</td>
<td>$\emptyset$</td>
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</tbody>
</table>

Let quota vector $q = (q_{a_1}, q_{a_2}) = (2, 1)$ and let $\beta = (b_1, b_2)$ be the order of the $B$-agents. One easily verifies that the (boxed) matching

$$
\mu : \begin{array} {c|c}
   a_1 & a_2 \\
   \emptyset & b_2 \\
\end{array}
$$

is stable, i.e., $\mu \in \Sigma(P)$. However, matching $\mu \notin \mathcal{O}_{\beta,q}(P)$. To see this, suppose $\mu \in \mathcal{O}_{\beta,q}(P)$. Let strategy profile $s \in \mathcal{E}_{\beta,q}(P)$ such that $\varphi_{\beta,q}(s) = \mu$. Let strategy $s'_{a_1} = \{b_1, b_2\}$ and strategy profile $s' = (s'_{a_1}, s_{-a_1})$. Then, at matching $\mu' \equiv \varphi_{\beta,q}(s')$ agent $a_1$’s match is $\mu'(a_1) = \{b_1, b_2\}$ which he strictly prefers to $\mu(a_1) = \emptyset$. Hence, $\mu \notin \mathcal{O}_{\beta,q}(P)$. □

Romero-Medina and Triossi (2014) study a many-to-one matching model where a set of students $S$ has to be matched to a set of colleges $C$. They assume that each student $s \in S$ finds it unacceptable to being matched to a set of two or more colleges (so, in particular each student $s$ has substitutable preferences). Romero-Medina and Triossi (2014) propose a mechanism, called the CSM (students apply Colleges Sequentially choose Mechanism), which coincides with our mechanism $\varphi_{\beta,q}$ by taking set $A = S$, set $B = C$, and setting for each agent $a \in A$, quota $q_a = 1$. Assuming furthermore that preferences $P_B$ are substitutable, Romero-Medina and Triossi (2014, Proposition 1) show that in this particular case the mechanism implements the set of stable matchings, i.e., it is possible to obtain the other inclusion in Theorem 1: $\Sigma(P) \supseteq \mathcal{O}_{\beta,q}(P)$.

**Proposition 1.** (Romero-Medina and Triossi, 2014, Proposition 1)

*For any $(\beta,q)$ and any preference profile $P$ where $P_B$ is substitutable and for all $a \in A$, $q_a = 1$,*

$$
\mathcal{O}_{\beta,q}(P) = \Sigma(P).
$$

The next two examples show that Proposition 1 of Romero-Medina and Triossi (2014) is tight in the sense that under a slight relaxation of the assumptions, implementation needs no longer be possible.

The first example related to Proposition 1 of Romero-Medina and Triossi (2014) shows that an unstable SPE outcome may exist if some $B$-agent has preferences that are not substitutable (even when all other preferences are substitutable and all quotas equal 1).
Example 2. (For some \( b \in B \), \( P_b \) is not substitutable, for all \( a \in A \), \( q_a = 1 \), and \( \Sigma(P) \not\subseteq O^{\beta,q}(P) \)) Consider the market with \( A = \{a_1, a_2\} \), \( B = \{b_1, b_2\} \), and preference profile \( P \) given by Table 2. Note that all preferences except for those of agent \( b_2 \) are substitutable. Hence, by Theorem 1, \( \Sigma(P) \subseteq O^{\beta,q}(P) \).

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Table 2: Preference profile \( P \) in Example 2

Let quota vector \( q = (q_{a_1}, q_{a_2}) = (1, 1) \) and let \( \beta = (b_1, b_2) \) be the order of the \( B \)-agents. We show that \( \Sigma(P) \not\subseteq O^{\beta,q}(P) \).

Let \( s \) be the strategy profile where \( s_{a_1} = \{b_1\} \), \( s_{a_2} = \{b_1, b_2\} \), and both \( B \)-agents choose optimally according to their preferences in all their decision nodes. One easily verifies that the (boxed) matching

\[
\mu : \begin{array}{ccc}
\ a_1 & a_2 \\
\ b_1 & \ | & \\
\ b_2 & \ | \\
\end{array}
\]

is the resulting matching, i.e., \( \mu = \varphi^{\beta,q}(s) \). We claim that strategy profile \( s \) is an SPE, i.e., \( s \in E^{\beta,q}(P) \). To see this, suppose there is a profitable deviation \( s'_{a_1} \) for agent \( a_1 \). Then, \( s'_{a_1} = \{b_1, b_2\} \) or \( s'_{a_1} = \{b_2\} \). However, in the first case, agent \( a_1 \) would again be matched to \( b_1 \). In the second case, agent \( a_1 \) would remain unmatched. Suppose now that there is a profitable deviation \( s'_{a_2} \) for agent \( a_2 \). Then, \( s'_{a_2} = \{b_1\} \) which however would leave agent \( a_2 \) unmatched. Thus, \( s \in E^{\beta,q}(P) \) and \( \mu \in O^{\beta,q}(P) \). But since \( (a_1, b_2) \) is a blocking pair for \( \mu \), \( \mu \) is not stable; i.e., \( \mu \not\in \Sigma(P) \). Hence, \( \Sigma(P) \not\subseteq O^{\beta,q}(P) \). \( \diamond \)

The second example related to Proposition 1 of Romero-Medina and Triossi (2014) shows that an unstable SPE outcome may exist if some \( A \)-agent has a quota that is larger than 1 (even when all preferences are substitutable and all other quotas equal 1).

Example 3. (\( P \) substitutable, for some \( a \in A \), \( q_a > 1 \), and \( \Sigma(P) \not\subseteq O^{\beta,q}(P) \)) Consider the market with \( A = \{a_1, a_2\} \), \( B = \{b_1, b_2\} \), and preference profile \( P \) given by Table 3. Note that all preferences are substitutable. Hence, by Theorem 1, \( \Sigma(P) \subseteq O^{\beta,q}(P) \).

Let quota vector \( q = (q_{a_1}, q_{a_2}) = (2, 1) \) and let \( \beta = (b_1, b_2) \) be the order of the \( B \)-agents. We show that \( \Sigma(P) \not\subseteq O^{\beta,q}(P) \).
Let $s$ be the strategy profile where $s_{a_1} = \{b_2\}$, $s_{a_2} = \{b_1, b_2\}$, and both $B$-agents choose optimally according to their preferences in all their decision nodes. One easily verifies that the (boxed) matching

$$
\mu : \begin{array}{ccc}
a_1 & a_2 \\
b_2 & b_1
\end{array}
$$

is the resulting matching, i.e., $\mu = \varphi^{\beta, q}(s)$. We claim that strategy profile $s$ is an SPE, i.e., $s \in E^{\beta, q}(P)$, To see this, note that agent $a_2$ gets his most preferred match and that $B$-agents choose optimally. Hence, $a_1$ is the only possible candidate for a profitable deviation. Suppose there is a profitable deviation $s'_{a_1}$ for agent $a_1$. Then, $s'_{a_1} = \{b_1\}$ or $s'_{a_1} = \{b_1, b_2\}$. However, in both cases one easily verifies that at strategy profile $s' = (s'_{a_1}, s_{-a_1})$ agent $a_1$ is matched to $\{b_1\}$. Hence, $s'_{a_1}$ is not a profitable deviation for agent $a_1$. Thus, $s \in E^{\beta, q}(P)$ and $\mu \in O^{\beta, q}(P)$. But since $(a_1, b_1)$ is a blocking pair for matching $\mu$, $\mu$ is not stable; i.e., $\mu \notin \Sigma(P)$. Hence, $\Sigma(P) \subsetneq O^{\beta, q}(P)$.

Example 3 shows that if some quota is larger than 1, then not all equilibrium outcomes need to be stable. We next show that if all quotas are large enough, then all equilibrium outcomes are guaranteed to be stable matchings (without any assumptions on the preferences!).

We say that quotas are non-binding if for all agents $a \in A$, $q_a \geq |B|$. When quotas are non-binding, at any strategy profile $s$, no $A$-agent ever becomes unavailable, i.e., if an agent $a$ decides to apply to set $s_a$, then any agent $b \in s_a$ can choose agent $a$ in any of its decision nodes.

Our second result shows that when quotas are non-binding, the $[A$ simultaneously apply – $B$ sequentially choose$]$ mechanism $\varphi^{\beta, q}$ implements in SPE a subset of the set of stable matchings.

**Theorem 2. (Non-binding quotas guarantee stability in equilibrium)**

*For any $(\beta, q)$ and any preference profile $P$ where quotas are non-binding,$^7$ $\mathcal{O}^{\beta, q}(P) \subseteq \Sigma(P)$.***

$^7$Alternatively, instead of requiring that quotas are non-binding, we could restrict $A$-agents’ strategies to not exceed their quotas: the result and the proof would then remain the same (but limiting the number of applications an $A$-agent can submit might be difficult to enforce in practice).
Proof. Let $P$ be a preference profile. Let matching $\mu$ be an SPE outcome, i.e., $\mu \in \mathcal{O}^{\beta,q}(P)$. Suppose matching $\mu$ is not stable, i.e., $\mu \not\in \Sigma(P)$. Let strategy profile $s \in \mathcal{E}^{\beta,q}(P)$ such that $\mu = \varphi^{\beta,q}(s)$. Since $\mu$ is an equilibrium outcome, it is individually rational. So, there is a blocking pair $(a,b) \in A \times B$ with $a \not\in \mu(b)$, $b \in \text{Ch}(\mu(a) \cup \{b\}, P_a)$, and $a \in \text{Ch}(\mu(b) \cup \{a\}, P_b)$.

Let strategy $s'_a = \text{Ch}(\mu(a) \cup \{b\}, P_a)$ and strategy profile $s' = (s'_a, s_{-a})$. We show that strategy $s'_a$ is a profitable deviation for agent $a$. Since $b \in \text{Ch}(\mu(a) \cup \{b\}, P_a)$, $\text{Ch}(\mu(a) \cup \{b\}, P_a) P_a \mu(a)$ and it suffices to show that at strategy profile $s'$ each agent in $s'_a$ chooses $a$.

Note that $s'_a \subseteq \mu(a) \cup \{b\}$. Hence, each agent in $s'_a \setminus \{b\}$ receives the same set of applications at strategy profile $s$ and at strategy profile $s'$. Since quotas are non-binding, at strategy profile $s'$ each agent in set $s'_a \setminus \{b\}$ chooses the same set of agents including agent $a$.

Next, we prove that $b \not\in s_a$. Suppose to the contrary that $b \in s_a$. Then, agent $a \in \{\bar{a} \in A : b \in s_a\}$. Since $\mu(b) = \text{Ch}(\{\bar{a} \in A : b \in s_a\}, P_b)$ it follows that $\mu(b) = \text{Ch}(\mu(b) \cup \{a\}, P_b)$. Thus, $a \in \text{Ch}(\mu(b) \cup \{a\}, P_b)$ implies $a \in \mu(b)$; a contradiction. So, $b \not\in s_a$.

Since $b \in s'_a \setminus s_a$ and strategy profile $s'$ only contains a unilateral deviation from strategy profile $s$, at strategy profile $s'$ agent $b$ receives the same set of applications as at strategy profile $s$ and in addition the application of $a$. In other words, $\{\bar{a} \in A : b \in s'_a\} = \{\bar{a} \in A : b \in s_a\} \cup \{a\}$. Suppose agent $b$ does not choose agent $a$ at strategy profile $s'$. Then, $a \not\in \text{Ch}(\{\bar{a} \in A : b \in s'_a\}, P_b) = \text{Ch}(\{\bar{a} \in A : b \in s_a\} \cup \{a\}, P_b) = \text{Ch}(\mu(b) \cup \{a\}, P_b)$, where the last equality follows from $a \not\in \mu(b) = \text{Ch}(\{\bar{a} \in A : b \in s_a\})$. Since we obtain a contradiction to $a \in \text{Ch}(\mu(b) \cup \{a\}, P_b)$, it follows that agent $b$ chooses agent $a$ at strategy profile $s'$. This shows that strategy $s'_a$ is a profitable deviation for agent $a$; a contradiction. \qed

The following result is a corollary to Theorems 1 and 2.

**Corollary 1. (Implementation)**

For any $(\beta,q)$ and any preference profile $P$ where $P_A$ is substitutable and quotas are non-binding,

$$\mathcal{O}^{\beta,q}(P) = \Sigma(P).$$

Corollary 1 subsumes results obtained by Romero-Medina and Triossi (2014, Proposition 2) and Sotomayor (2003, Theorems 1 and 2). More specifically, Romero-Medina and Triossi (2014) consider a mechanism, called the SSM (colleges apply Students Sequentially choose Mechanism), where colleges first simultaneously propose to students and then students sequentially pick a college. The SSM coincides with our mechanism $\varphi^{\beta,q}$ by taking set $A = C$, set $B = S$, and setting for each $a \in A$, $q_a = |B|$.

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8Essentially, the second phase of the SSM is equivalent to a simultaneous-move game among students. Games in which first colleges move simultaneously and then students move simultaneously are also studied in Alcalde and Romero-Medina (2000). As a consequence, Proposition 2 in Romero-Medina and Triossi (2014) is closely related to Theorem 4.1 in Alcalde and Romero-Medina (2000).
Corollary 2. (Romero-Medina and Triossi, 2014, Proposition 2)
For any \((\beta, q)\) and any preference profile \(P\) where \(PA\) is substitutable, for all agents \(b \in B\) and for all \(T \subseteq A, |T| \geq 2 \Rightarrow \emptyset P_b T\), and for all agents \(a \in A, q_a = |B|,\)

\[
\mathcal{O}^{\beta,q}(P) = \Sigma(P).
\]

4 Concluding remark: setwise stability

For many-to-many matching markets, the following stronger stability notion is also often considered. Let \(P\) be a preference profile. Then, matching \(\mu\) is blocked by a set (of agents) \(I' = A' \cup B' \subseteq A \cup B, I' \neq \emptyset\), if there exists a matching \(\mu'\) such that (a) for all \(i \in I, \mu'(i) \setminus \mu(i) \subseteq I'\) — new matches are among the members of the blocking coalition only— and (b) for all \(i \in I', \mu'(i) P_i \mu(i)\) and \(\mu'(i) = \text{Ch}(\mu'(i), P_i)\) — all members of the blocking coalition receive a better and individually rational match. Note that agents outside the blocking coalition are not matched to new agents, but possibly some of their matches are canceled by members of the blocking coalition. A matching \(\mu\) is setwise stable if it is individually rational and not blocked by any set of agents \(I' = A' \cup B'\). Let \(\Omega(P)\) denote the set of setwise stable matchings.

First, note that a setwise stable matching is always (pairwise) stable, i.e., for all preference profiles \(P\), \(\Omega(P) \subseteq \Sigma(P)\). Hence, Theorem 1 would also hold if we used setwise stability instead of (pairwise) stability.

Second, we show that a result similar to Theorem 2 cannot be obtained if we used setwise stability instead of (pairwise) stability.

Example 4. (Setwise stability not obtained in equilibrium)
Consider the market introduced by Blair (1988, Example 2.6) where \(A = \{a_1, a_2, a_3\}, B = \{b_1, b_2, b_3\}\), and preference profile \(P\) is given by Table 4. Note that all preferences are substitutable.

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Table 4: Preference profile \(P\) in Example 4

Let quota vector \(q = (q_{a_1}, q_{a_2}, q_{a_3}) = (2, 2, 2)\) and let \(\beta = (b_1, b_2, b_3)\) be the order of the \(B\)-agents. We show that \(\Omega(P) \subsetneq \mathcal{O}^{\beta,q}(P)\). First, Blair (1988) shows that even though a unique stable (boxed) matching
agents simultaneously apply and Corollary 1 need not hold. Interestingly, for the variation of our mechanisms where A and non-binding quotas, an implementation result for setwise Example 4 shows that in many-to-many matching markets with substitutable preferences. Thus, \( \Sigma(P) = \{ \mu \} \) and \( \Omega(P) = \emptyset \).

Next, we show that matching \( \mu \) is an SPE outcome, i.e., \( \mu \in O^{\beta,q}(P) \). Let \( s \) be the strategy profile where \( s_{a_1} = \{b_1\}, s_{a_2} = \{b_2\}, s_{a_3} = \{b_3\} \), and all B-agents choose optimally according to their preferences in all their decision nodes. One easily verifies that the (boxed) matching \( \mu \) is the resulting matching, i.e., \( \mu = \varphi^{\beta,q}(s) \).

We claim that strategy profile \( s \) is an SPE, i.e., \( s \in E^{\beta,q}(P) \). To see this, suppose there is a profitable deviation \( s'_{a_1} \) for agent \( a_1 \). Then, \( s'_{a_1} = \{b_1, b_2\} \), \( s'_{a_1} = \{b_2, b_3\} \), or \( s''_{a_1} = \{b_1, b_2, b_3\} \). However, in the first case, agent \( a_1 \) would again be matched with \( \{b_1\} \), in the second case, agent \( a_1 \) would be matched with \( \{b_3\} \), and in the third case, agent \( a_1 \) would be matched with \( \{b_1, b_3\} \). Hence, \( s'_{a_1} \) is not a profitable deviation for agent \( a_1 \). Similarly, we can show that neither agent \( a_2 \) nor agent \( a_3 \) has a profitable deviation. Thus, \( s \in E^{\beta,q}(P) \) and \( \mu \in O^{\beta,q}(P) \). Hence, \( \Omega(P) \subseteq O^{\beta,q}(P) \).

Note that Example 4 remains valid with non-binding quotas, e.g., \( q = (3,3,3) \). Thus, Example 4 shows that in many-to-many matching markets with substitutable preferences and non-binding quotas, an implementation result for setwise stable matchings similar to Corollary 1 need not hold. Interestingly, for the variation of our mechanisms where A-agents simultaneously apply and B-agents simultaneously choose, Echenique and Oviedo (2006, Corollary 7.2) show that the set of setwise stable matchings can be implemented if A-agents have substitutable preferences and B-agents have so-called strongly substitutable preferences. The preferences \( P_i \) of an agent \( i \in I \) are strongly substitutable if for all \( j \in T_i \) and for all sets \( T', T \subseteq T_i \) with \( T' P T, [j \in Ch(T' \cup \{j\}, P_i) \text{ implies } j \in Ch(T \cup \{j\}, P_i)] \). In Example 4, all agents’ preferences are substitutable but not strongly so.\(^9\) Because in our setting the effect of B-agents moving simultaneously can be obtained via non-binding quota, an implication of Echenique and Oviedo (2006, Corollary 7.2) is the following corollary.

**Corollary 3. (Implementation)**

For any \((\beta,q)\) and any preference profile \( P \) where \( P_A \) is substitutable, \( P_B \) is strongly substitutable, and quotas are non-binding,

\[ O^{\beta,q}(P) = \Omega(P). \]

\(^9\)For instance, \( P_{b_1} \) violates strong substitutability since \( T' \equiv \{a_2, a_3\} P_{b_1} \{a_1\} \equiv T \) and \( a_3 \in Ch(T' \cup \{a_3\}, P_{b_1}) \), but \( a_3 \notin Ch(T \cup \{a_3\}, P_{b_1}) \).

11
References


