Asset Prices, Liquidity, and Monetary Policy
in an Exchange Economy

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Abstract
I formulate a search-based asset-pricing model where equity shares and fiat money can be used as means of payment. I characterize a family of optimal stochastic monetary policies. Every policy in this family implements Friedman’s prescription of zero nominal interest rates. Under an optimal policy, equity prices and returns are independent of monetary considerations. I also study a perturbation of the family of optimal policies that targets a constant but nonzero nominal interest rate. Under such policies, the average real return on equity is negatively correlated with the average inflation rate.

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1 Introduction

Many financial assets are to some degree, valued not only as claims to streams of consumption goods, but also for their liquidity— their usefulness in the mechanism of exchange. This observation materializes in its purest form with fiat money: an asset that is universally used in exchange and sells at a positive price, even though it represents a formal claim to nothing. In this paper I formulate a search-based model where money and an asset that represents a claim to a stochastic stream of real dividends (an equity share) can be used as means of payment, and use the theory to derive the asset-pricing implications of monetary policy.

In Section 2 I present the basic model. In Section 3 I show how liquidity considerations affect equity prices and returns in an economy with no money. I find that if the asset can help relax trading constraints in some state of the world, the equilibrium asset price is higher and its measured rate of return (dividend yield plus capital gains) is lower than they would be in an economy with no liquidity needs.

In Section 4 I introduce fiat money and define a recursive monetary equilibrium. In Section 5 I characterize a class of optimal monetary policies, and describe the behavior of asset prices, asset returns, output, inflation, and the nominal interest rate under the optimal policy. Every policy in this family implements Friedman’s prescription of zero nominal interest rates. Under an optimal policy, equity prices and returns are independent of monetary considerations.

In Section 6, I consider perturbations of the optimal monetary policy that consist of targeting a constant nominal interest rate, and discuss some of the positive implications of changes in the nominal interest rate or the inflation rate on equity prices, equity returns, and output. I find that the price of equity is increasing, and real balances are decreasing in the nominal interest rate target. The analysis also provides insights on how monetary policy must be conducted in order to support a recursive monetary equilibrium with a constant nominal interest rate (with the Pareto optimal equilibrium in which the nominal rate is zero as a special case): The growth rate of the money supply must be relatively low in states in which the real value of the equilibrium equity holdings is below average. Something similar happens with the implied inflation rate: it is relatively low between state $x$ and a next-period state $x'$, if the realized real value of the equilibrium equity holdings in state $x'$ is below its state-$x$ conditional expectation. I also find that on average, liquidity considerations can introduce a negative relationship between the nominal interest rate (and the inflation rate) and equity returns. If the average rate of inflation
is higher, real money balances are lower, and the liquidity return on equity rises, which causes its price to rise and its real measured rate of return to fall.

This paper is related to a large literature that studies how monetary considerations may help explain various features of asset prices. Some examples include Balduzzi (1996), Bansal and Coleman (1996), Bohn (1991), Boyle and Young (1988), Danthine and Donaldson (1986), Giovannini and Labadie (1991), Kiyotaki and Moore (2005), Svensson (1985), and Townsend (1987). The approach I follow is different from these previous studies. First, these papers assume that money plays a special role, either because it is the only financial asset that satisfies a cash-in-advance constraint, or because it is the only financial asset that enters the agents’ utility functions. In contrast, I do not assume that money plays a special role in exchange. Second, my work builds on the literature that provides micro foundations for monetary economics based on search theory, as pioneered by Kiyotaki and Wright (1989). Specifically, the model is a version of Lagos and Wright (2005), augmented to allow for aggregate liquidity shocks, and another financial asset that can be used as means of payment the same way money can.1

2 The model

Time is discrete, and the horizon infinite. There is a [0, 1] continuum of infinitely lived agents. Each time period is divided into two subperiods where different activities take place. There are three nonstorable and perfectly divisible consumption goods at each date: fruit, general goods, and special goods.2 Fruit and general goods are homogeneous goods, while special goods come in many varieties. The only durable commodity in the economy is a set of “Lucas trees.” The number of trees is fixed and equal to the number of agents. Trees yield (the same amount of) a random quantity \( d_t \) of fruit in the second subperiod of every period \( t \). The realization of \( d_t \) becomes known to all at the beginning of period \( t \) (when agents enter the first subperiod).

1 Lagos and Rocheteau (2008) was the first paper to extend Lagos and Wright (2005) to allow for another asset that competes with money as a medium of exchange. Lagos (2006) formulates a real version of Lagos and Wright (2005) with aggregate uncertainty, in which equity shares and government bonds can serve as means of payment, and uses it to study the equity premium and the risk-free rate puzzles. Ravikumar and Shao (2006) consider a related model that combines features of Lucas (1978) with features of Lagos and Wright (2005) and Shi (1997) to study the excess volatility puzzle. Geromichalos et al (2007) consider a version of Lagos and Rocheteau (2008) in which the real asset that competes with money is in fixed supply. Lester et al (2008) consider a version of Geromichalos et al (2007) in which money can be used as means of payment in all bilateral meetings, while the real asset is only accepted in some bilateral meetings.

2 “Nonstorable” means that the goods cannot be carried from one subperiod to the next. This formulation with three consumption goods allows a parsimonious integration of the asset pricing model of Lucas (1978) with the model of exchange in Lagos and Wright (2005).
Production of fruit is entirely exogenous: no resources are utilized and it is not possible to affect the output at any time. The motion of \( dt \) will be taken to follow a Markov process, defined by its transition function \( F(x',x) = \Pr (d_{t+1} \leq x'|d_t = x) \), where \( F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) is continuous. For each fixed \( x \), \( F(\cdot,x) \) is a distribution function with support \( \Xi \subseteq (0,\infty) \).

Assume that the process defined by \( F \) has a stationary distribution \( \psi(\cdot) \), the unique solution to \( \psi(x_0) = \int F(x',x) d\psi(x) \), and that \( F \) has the Feller property, i.e., for any continuous real-valued function \( g \) on \( \Xi \), \( \int g(x')dF(x',x) \) is a continuous function of \( x \).

In each subperiod, every agent is endowed with \( \bar{n} \) units of time which can be employed as labor services. In the second subperiod, each agent has access to a linear production technology that transforms labor services into general goods. In the first subperiod, each agent has access to a linear production technology that transforms his own labor input into a particular variety of the special good that he himself does not consume. This specialization is modeled as follows. Given two agents \( i \) and \( j \) drawn at random, there are three possible events. The probability that \( i \) consumes the variety of special good that \( j \) produces but not vice-versa (a single coincidence) is denoted \( \alpha \). Symmetrically, the probability that \( j \) consumes the special good that \( i \) produces but not vice-versa is also \( \alpha \). In a single-coincidence meeting, the agent who wishes to consume is the buyer, and the agent who produces, the seller. The probability neither wants anything the other can produce is \( 1 - 2\alpha \), with \( \alpha \leq 1/2 \). In contrast to special goods, fruit and general goods are homogeneous, and hence consumed (and in the case of general goods, also produced) by all agents.

In the first subperiod, agents participate in a decentralized market where trade is bilateral (each meeting is a random draw from the set of pairwise meetings), and the terms of trade are determined by bargaining. The specialization of agents over consumption and production of the special good combined with bilateral trade, give rise to a double-coincidence-of-wants problem in the first subperiod. In the second subperiod, agents trade in a centralized market. Agents cannot make binding commitments, and trading histories are private in a way that precludes any borrowing and lending between people, so all trade—both in the centralized and decentralized markets—must be quid pro quo.

Each tree has outstanding one durable and perfectly divisible equity share that represents the bearer’s ownership of a tree and confers him the right to collect the fruit dividends. I will later introduce another perfectly divisible asset—fiat money. All assets are perfectly recognizable, cannot be forged, and can be traded among agents both in the centralized and decentralized
markets. At $t = 0$ each agent is endowed with $a_0^s$ equity shares (and possibly also $a_0^m$ units of fiat money).

Let the utility function for special goods, $u : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, and the utility function for fruit, $U : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, be continuously differentiable, bounded by $B$ on $\Xi$, increasing, and strictly concave, with $u(0) = U(0) = 0$. Let $-n$ be the utility from working $n$ hours in the first subperiod. Also, suppose there exists $q^* \in (0, \infty)$ defined by $u'(q^*) = 1$, with $q^* \leq \bar{n}$. Let both, the utility for general goods, and the disutility from working in the second subperiod, be linear. The agent’s preferences are:

$$\liminf_{T \to \infty} E_0 \left\{ \sum_{t=0}^{T} \beta^t \left[ u(q_t) - n_t + U(c_t) + y_t - h_t \right] \right\},$$

where $\beta \in (0, 1)$, $q_t$ and $n_t$ are the quantities of special goods consumed and produced in the decentralized market, $c_t$ denotes consumption of fruit, $y_t$ consumption of general goods, $h_t$ the hours worked in the second subperiod, and $E_t$ is an expectations operator conditional on the information available to the agent at time $t$, defined with respect to the matching probabilities and the probability measure induced by $F$.

### 3 Asset prices and liquidity in a real economy

I begin by considering a real economy where the equity share is the only asset. Let $W(a_t, d_t)$ denote the value function of an agent who enters the centralized market holding $a_t$ shares in a period when dividends are $d_t$, and let $V(a_t, d_t)$ denote the corresponding value when he enters the decentralized market. These value functions satisfy the following Bellman equation:

$$W(a_t, d_t) = \max_{c_t, y_t, h_t, a_{t+1}} \left\{ U(c_t) + y_t - h_t + \beta E(V(a_{t+1}, d_{t+1})) \right\}$$

s.t. $c_t + w_t y_t + \phi_t a_{t+1} = (\phi_t + d_t) a_t + w_t h_t$

$$0 \leq c_t, 0 \leq h_t \leq \bar{n}, 0 \leq a_{t+1}.$$

The agent chooses consumption of fruit ($c_t$), consumption of general goods ($y_t$), hours of work devoted to production of general goods ($h_t$), and an end-of-period portfolio ($a_{t+1}$). Dividends are paid to the bearer of the equity share after decentralized trade, but before the time-$t$
centralized trading session. Fruit is used as numéraire: \( w_t \) is the relative price of general goods, and \( \phi_t \) is the (ex-dividend) price of a share. Substitute the first two constraints into the objective, let \( \lambda_t \equiv \frac{1}{w_t} (\phi_t + d_t) \), and rearrange to arrive at:

\[
W(a_t, d_t) = \lambda_t a_t + \max_{c_t} \left[ U(c_t) - \frac{c_t}{w_t} \right] + \max_{a_{t+1}} \left[ -\frac{\phi_t a_{t+1}}{w_t} + \beta EV(a_{t+1}, d_{t+1}) \right]. \tag{1}
\]

When a buyer and a seller have share holdings \( a \) and \( \tilde{a} \) respectively, the terms at which they trade in the decentralized market are \( [q(a, \tilde{a}), p(a, \tilde{a})] \), where \( q(a, \tilde{a}) \in \mathbb{R}_+ \) is the quantity of special good traded, and \( p(a, \tilde{a}) \in \mathbb{R}_+ \) represents the transfer of assets from the buyer to the seller. The value of an agent who enters the decentralized market with share holdings \( a \) in a period when the dividend realization is \( d \), satisfies

\[
V(a, d) = \alpha \int \{u[q(a, \tilde{a})] + W[a - p(a, \tilde{a}), d]\} dG(\tilde{a}) + \\
\alpha \int \{-q(\tilde{a}, a) + W[a + p(\tilde{a}, a), d]\} dG(\tilde{a}) + (1 - 2\alpha) W(a, d), \tag{2}
\]

where \( G \) denotes the distribution of share holdings among the population.

Consider a meeting in the decentralized market between a buyer who holds \( a_t \) and a seller who holds \( \tilde{a}_t \). The terms of trade \((q_t, p_t)\) are determined by Nash bargaining where the buyer has all the bargaining power.\(^4\) Thus, \((q_t, p_t)\) solves

\[
\max_{q_t, p_t \leq a_t} \left[ u(q_t) + W(a_t - p_t, d_t) - W(a_t, d_t) \right] \text{ s.t. } -q_t + W(\tilde{a}_t + p_t, d_t) \geq W(\tilde{a}_t, d_t).
\]

The constraint \( p_t \leq a_t \) indicates that the buyer cannot spend more assets than he owns. Since \( W(a_t + p_t, d_t) - W(a_t, d_t) = \lambda_t p_t \), the bargaining problem reduces to:

\[
\max_{q_t, p_t \leq a_t} \left[ u(q_t) - \lambda_t p_t \right] \text{ s.t. } -q_t + \lambda_t p_t \geq 0.
\]

If \( \lambda_t a_t \geq q^* \), the buyer exchanges \( p_t = q^*/\lambda_t \leq a_t \) of his shares for \( q^* \) special goods. Else, he gives the seller all his shares, i.e., \( p_t = a_t \), in exchange for \( q_t = \lambda_t a_t \) special goods. Hence, the quantity of special goods traded is \( q(\lambda_t a_t) \), where

\[
q(x) = \min(x, q^*). \tag{3}
\]

With the bargaining solution and the linearity of \( W(a, d) \), the value of search (2) becomes

\[
V(a, d) = S(\lambda a) + W(a, d), \tag{4}
\]

\(^4\)See Lagos (2006) for an analysis with generalized Nash bargaining in a related model.
where
\[ S(x) = \alpha \{ u[q(x)] - q(x) \} \]  \hspace{1cm} (5)

is the expected gain from trading in the decentralized market. Observe that \( S \) is twice differentiable almost everywhere, with \( S'(x) \geq 0 \) and \( S''(x) \leq 0 \) (both inequalities are strict for \( x < q^* \)). Having characterized the terms of trade in decentralized exchange, I turn to the agent’s individual optimization problem in the centralized market.

The agent’s problem in the second subperiod is summarized by (1). Given that \( U \) is strictly concave, the optimal consumption of fruit satisfies
\[ w_t U'(c_t) = 1, \]  \hspace{1cm} (6)

and the first-order necessary and sufficient condition for the choice of \( a_{t+1} \) is
\[ U'(c_t) \phi_t = \beta E_t V_1 (a_{t+1}, d_{t+1}). \]

From (4), \( V_1 (a_{t+1}, d_{t+1}) = [1 + \alpha (u'[q(\lambda_{t+1} a_{t+1})] - 1)] \lambda_{t+1}, \) and \( V_{11} (a_{t+1}, d_{t+1}) \leq 0 \ (< 0 \) for \( \lambda_{t+1} a_{t+1} < q^* \)). None of the agent’s choices depend on his individual asset holdings, so as in Lagos and Wright (2005), \( G \) will be degenerate (at the mean) in equilibrium.\(^5\)

I will consider an equilibrium in which all prices are time-invariant functions of the aggregate state, \( d_t \): \( w_t = w(d_t), \phi_t = \phi(d_t), \) and therefore, \( \lambda_t = \frac{1}{w(d_t)} [\phi(d_t) + d_t] =: \lambda(d_t). \) In words, \( \lambda(d_t) \) is the (cum-dividend) price of an equity share in state \( d_t \), in expressed terms of general goods. With (6),
\[ \lambda(x) = U'(x) [\phi(x) + x]. \]

**Definition 1** A recursive equilibrium for the economy with equity is a collection of individual decision rules \( c_t = c(d_t), a_{t+1} = a(d_t), \) pricing functions \( w_t = w(d_t) \) and \( \phi_t = \phi(d_t), \) and bilateral terms of trade \( q_t = q(d_t) \) and \( p_t = p(d_t) \) such that: (i) given prices and the bargaining protocol, the decision rules \( c(\cdot), \) and \( a(\cdot), \) solve the agent’s problem in the centralized market; (ii) the terms of trade in a bilateral meeting where the buyer holds \( a, \) are determined by Nash bargaining, i.e., \( q(d_t) = q[\lambda(d_t) a], \) and \( p(d_t) = q[\lambda(d_t) a]/\lambda(d_t); \) and (iii) prices are such that the centralized market clears: \( c(d_t) = d_t, \) and \( a(d_t) = 1 \) for all \( d_t. \)

\(^5\)Also, if \( \lambda_{t+1} a_{t+1} < q^* \) for some realizations of the dividend process, the portfolio choice problem at date \( t \) has a unique solution, implying that the distribution of assets must be degenerate at the beginning of each decentralized round of trade. Regarding the constraints, the agent’s maximization is subject to \( 0 \leq c_t, \) which will not bind if, for example, \( U'(0) = +\infty. \) Similarly, in equilibrium, shares will be valued and somebody has to hold them, so \( 0 \leq a_{t+1} \) will not bind either.
The Euler equation for equity holdings implies the pricing function for equity shares satisfies

\[ U' (x) \phi (x) = \beta \int L [\phi (x')] U' (x') \left[ \phi (x') + x' \right] dF (x', x), \]

where

\[ L [\lambda (x')] \equiv 1 - \alpha + \alpha u' (q [\lambda (x')]). \]

This can be written as

\[ U' (x) \phi (x) = \beta \int_\Omega \{ 1 - \alpha + \alpha u' [\lambda (x')] \} U' (x') \left[ \phi (x') + x' \right] dF (x', x) + \]

\[ \beta \int_{\Omega^c} U' (x') \left[ \phi (x') + x' \right] dF (x', x), \]

where

\[ \Omega = \{ x \in \Xi : \lambda (x) < q^* \} \]

and \( \Omega^c \) denotes its complement. The set \( \Omega \) contains the realizations of the aggregate dividend process for which the asset has value for its role as a medium of exchange, in addition to its “intrinsic” value, i.e., that which stems from the right that ownership of the asset confers to collect future dividends. So there is a sense in which \( L \) in (7) can be thought of as a stochastic liquidity factor. Notice that equation (8) reduces to equation (6) in Lucas (1978) if either \( \Omega = \emptyset \) (the asset has no liquidity value in any state of the world, i.e., \( L [\lambda (x)] = 1 \) for all \( x \)), or \( \alpha = 0 \) (agents have no liquidity needs). In what follows, it will often prove convenient to express (7) as a functional equation in \( \lambda \):

\[ \lambda (x) = \beta \int L [\lambda (x')] \lambda (x') dF (x', x) + xU' (x). \]

In applications, one will typically have to solve (9) numerically, but some useful insights regarding the properties of \( \phi (x) \) and the structure of the set \( \Omega \), can be gained by considering some special cases that can be solved by paper-and-pencil methods.\(^6\)

### 3.1 i.i.d. returns

Suppose \( \{d_t\} \) is a sequence of independent random variables: \( F (d_{t+1}, d_t) = F (d_{t+1}) \). In this case, (9) implies \( \lambda (x) - xU' (x) = \beta \Delta \), where \( \Delta \) satisfies

\[ \Delta = \int [1 - \alpha + \alpha u' (q [\beta \Delta + zU' (z)])] [\beta \Delta + zU' (z)] dF (z). \]

\(^6\)Notice that for the case of \( \alpha = 0 \) analyzed by Lucas (1978), it is straightforward to show that (9) is a contraction. It appears that this may not the case for \( \alpha \in (0, 1] \), unless some more restrictive assumptions are made on \( u \).
Lemma 1 There exists a unique $\Delta$ that solves (10). This solution is positive, and strictly increasing in $\alpha$.

Given the value of $\Delta$ characterized by (10), the equity price function is

$$\phi(x) = \frac{\beta \Delta}{U'(x)},$$

(11)

and the set of realizations of the dividend process for which there is a liquidity return is

$$\Omega = \{x \in \Xi : xU'(x) < q^* - \beta \Delta\}.$$  

(12)

The following result provides a more detailed characterization of the set $\Omega$.

Claim 1 Assume $\Xi = [x, \infty)$, with $x > 0$, and let $\rho(x) \equiv \frac{-xU''(x)}{U'(x)}$.

(i) If $q^* \leq \frac{\beta}{1-\beta} \int zU'(z) dF(z)$, then $\Omega = \emptyset$.

(ii) If $q^* > \frac{\beta}{1-\beta} \int zU'(z) dF(z)$, and $\rho(x) > 1$ for all $x$, then:

(a) $\Omega = \emptyset$ if $q^* - \beta \Delta \leq \lim_{x \to \infty} xU'(x)$

(b) $\Omega = \{x \in \Xi : x > x^*\}$ if $\lim_{x \to \infty} xU'(x) < q^* - \beta \Delta \leq \underline{xU'}(x)$, where $x^*$ is the unique solution to $x^*U'(x^*) = q^* - \beta \Delta$

(c) $\Omega = \Xi$ if $\underline{xU'}(x) < q^* - \beta \Delta$

(iii) If $q^* > \frac{\beta}{1-\beta} \int zU'(z) dF(z)$, and $\rho(x) < 1$ for all $x$, then:

(a) $\Omega = \Xi$ if $\lim_{x \to \infty} xU'(x) \leq q^* - \beta \Delta$

(b) $\Omega = \{x \in \Xi : x < x^*\}$ if $\underline{xU'}(x) < q^* - \beta \Delta < \lim_{x \to \infty} xU'(x)$, where $x^*$ is the unique solution to $x^*U'(x^*) = q^* - \beta \Delta$

(c) $\Omega = \emptyset$ if $q^* - \beta \Delta \leq \underline{xU'}(x)$

(iv) If $q^* > \frac{\beta}{1-\beta} \int zU'(z) dF(z)$, and $\rho(x) = 1$ for all $x$, then $\Omega = \emptyset$ if $q^* \leq \frac{1}{1-\beta}$, and $\Omega = \Xi$ if $q^* > \frac{1}{1-\beta}$.

One can think of $q^*$ as indexing the economy’s liquidity needs. For instance, if $q^* \leq \frac{\beta}{1-\beta} \int zU'(z)$, then $\Omega = \emptyset$, and $\Delta = \frac{1}{1-\beta} \int zU'(z)$. That is, if $q^*$ is relatively low, then
asset prices reduce to those in the i.i.d. example in Lucas (1978). Clearly, the same happens if one simply specifies that the asset is completely illiquid, say by setting $\alpha = 0$.

In general, the *gross one-period return to equity* between any two states $x_i$ and $x_j$ is defined as $R^e(x_j, x_i) = \frac{\phi(x_j) + x_j}{\phi(x_i)}$. For the i.i.d. case,

$$R^e(x_j, x_i) = \left[1 + \frac{x_j U'(x_j)}{\beta \Delta} \right] \frac{U'(x_i)}{U'(x_j)}.$$  \hspace{1cm} (13)

Lemma 1 shows that $\Delta$ is increasing in $\alpha$, so as the probability the asset can be used in exchange rises, (11) indicates that the price of equity rises, and (13) that its (state-by-state) return falls.

Part (iv) of Claim 1 shows that the case of $\rho(x) = 1$ for all $x$, is particularly simple: the asset either provides liquidity in every state or in no state, and the latter is the case if $q^* \leq \frac{1}{1-\beta}$, which implies $\Delta = \frac{1}{1-\beta}$, and therefore $\phi(x) = \beta_0 x$. Conversely, if $\frac{1}{1-\beta} < q^*$, then $\Omega = \Xi$, and $\phi(x) = \beta \Delta x$, where $\Delta$ solves $u'(1 + \beta \Delta) = 1 + \frac{\lambda_0}{\lambda_1 + \beta \Delta j}$. It is easy to show that the solution satisfies $\frac{1}{1-\beta} < \Delta < \frac{q^* - 1}{\beta}$. The first inequality means that asset prices are higher in every state in the economy with liquidity needs (the economy with high $q^*$). The liquidity factor is constant in all states: $L = 1 + \alpha \left\{ u'(\min(1 + \beta \Delta, q^*)) - 1 \right\}$, and $L > 1$ since $\Delta < \frac{q^* - 1}{\beta}$. In this case it is also possible to show that the liquidity factor, $L$, is increasing in $\alpha$.

### 3.2 Correlated returns with log preferences over special goods

Next, generalize the dividend process by allowing it to be serially correlated over time, but specialize preferences over special goods by assuming $u(q) = \log q$.\footnote{Strictly speaking, standard CRRA preferences do not satisfy the maintained assumption $u(0) = 0$. But as in Lagos and Wright (2005), similar results would obtain by adopting $u(q) = \frac{q + b}{b} - \frac{1}{\sigma + b q}$ with $\sigma > 0$, and $b > 0$ but small. Note that $\frac{u'(q)}{u(q)} = \frac{\sigma}{1 + \sigma q}$, and as $\sigma \to 1$, $u(q) \to \ln(q + b) - \ln(b)$, and $\frac{u''(q)}{u(q)} \to \frac{1}{1 + b q}$.} In this case, $q^* = 1$, so $u'(q \lambda(x))] = \max[1, \lambda(x)^{-1}]$ and (9) becomes

$$\lambda(x) = \beta \int \left\{ (1 - \alpha) \lambda(x') + \alpha \max[\lambda(x'), 1] \right\} dF(x', x) + x U'(x).$$  \hspace{1cm} (14)

**Lemma 2** There exists a unique continuous and bounded function, $\lambda$, that solves (14). Moreover, $\lambda(x) > 0$ for all $x$.

In general, the liquidity constraint $\lambda(x) \leq 1$ may bind in some states and not in others, but to illustrate, consider two special cases. First, if the constraint never binds, i.e., $\lambda(x) \geq 1$ for all $x \in \Xi$, then (14) reduces to

$$\lambda(x) = \beta \int \lambda(x') dF(x', x) + x U'(x).$$  \hspace{1cm} (15)
which is identical to equation (6) in Lucas (1978), after substituting $\lambda(x) = U'(x)[\phi(x) + x]$. Alternatively, if the constraint binds in every state of the world, i.e., $\lambda(x) < 1$ for all $x \in \Xi$, then (14) becomes

$$\lambda(x) = \beta(1 - \alpha) \int \lambda(x')dF(x', x) + \beta\alpha + xU'(x).$$

Let $x = x_t$, $x' = x_{t+1}$, $\phi(x) = \phi_t$, revert to a sequential formulation and iterate on (16), to arrive at

$$\phi_t = \frac{\alpha\beta}{1 - \beta(1 - \alpha)} \frac{1}{U'(x_t)} + E_t \sum_{j=1}^{\infty} \frac{[\beta(1 - \alpha)]^j U'(x_{t+j})}{U'(x_t)} x_{t+j}. \quad (17)$$

If one shuts down the decentralized market, say by setting $\alpha = 0$, then (17) reduces to a standard textbook asset pricing equation (e.g., equation (3.11) in Sargent (1987), p. 96). Note that (17) was derived under the assumption that $\lambda(x_t) < 1$ for all $x_t$, or equivalently, that $U'(x_t)(\phi_t + x_t) < 1$ for all $x_t$. This is indeed the case in equilibrium if

$$\frac{\alpha\beta}{1 - \beta(1 - \alpha)} + E_t \sum_{j=1}^{\infty} [\beta(1 - \alpha)]^j U'(x_{t+j}) x_{t+j} < 1, \quad \text{for all } x_t. \quad (18)$$

For a particular specification of preferences, the following result provides a more explicit characterization of the set $\Omega$ under correlated returns.

**Claim 2** Suppose $u(c) = \log c$ and $U(c) = \varepsilon u(c)$. If $\varepsilon < 1 - \beta$, then $\Omega = \Xi$, and

$$\phi(x) = \frac{\beta[\alpha + (1 - \alpha)\varepsilon]}{\varepsilon[1 - \beta(1 - \alpha)]} x. \quad (19)$$

Alternatively, if $\varepsilon \geq 1 - \beta$, then $\Omega \subset \Xi$ (the asset provides no liquidity in some state).

4 **Asset prices and liquidity in a monetary economy**

Consider the economy analyzed in the previous section, but suppose there is a second asset: money. Money is intrinsically useless (it is not an argument of any utility or production function), and unlike equity, ownership of money does not constitute a right to collect any resources. Let $s_t = (d_t, M_t)$ denote the aggregate state of the economy at time $t$, where $M_t$ is the money supply at time $t$. The money supply is set by a government that injects or withdraws money via lump-sum transfers or taxes in the second subperiod of every period, i.e., $M_{t+1} = M_t + T_t$, where $T_t$ is the lump-sum transfer (or tax, if negative). Let $a = (a^s, a^m)$ denote the portfolio of
an agent who holds $a^s$ shares and $a^m$ dollars. Let $W(a, s)$ and $V(a, s)$ be the values from entering the centralized, and decentralized market, respectively, with portfolio $a$ when the aggregate state is $s$. These value functions satisfy the following Bellman equation:

$$W(a_t, s_t) = \max_{c_t,y_t,h_t,a_{t+1}} \{ U(c_t) + y_t - h_t + \beta EV(a_{t+1}, s_{t+1}) \}$$

s.t. $c_t + w_ty_t + \phi_t a_{t+1} = (\phi^s_t + d_t) a^s_t + \phi^m_t (a^m_t + T_t) + w_th_t$

$$0 \leq c_t, 0 \leq h_t \leq \bar{n}, 0 \leq a_{t+1},$$

where $\phi_t \equiv (\phi^s_t, \phi^m_t)$. The agent chooses consumption of fruit ($c_t$), consumption of general goods ($y_t$), labor supply ($h_t$), and an end-of-period portfolio ($a_{t+1}$). Again fruit is used as numéraire: $w_t$ is the relative price of the general good, $\phi^s_t$ is the (ex-dividend) price of a share, and $1/\phi^m_t$ the dollar price of fruit. Substitute the budget constraint into the objective and rearrange to arrive at:

$$W(a_t, s_t) = \lambda_t a_t + \tau_t + \max_{c_t} \left[ U(c_t) - \frac{c_t}{w_t} \right] + \max_{a_{t+1}} \left[ -\frac{\phi_t a_{t+1}}{w_t} + \beta EV(a_{t+1}, s_{t+1}) \right],$$  (20)

where $\tau_t = \frac{1}{w_t} \phi^m_t T_t$, and $\lambda_t = (\lambda^s_t, \lambda^m_t)$, with $\lambda^s_t \equiv \frac{1}{w_t} (\phi^s_t + d_t)$ and $\lambda^m_t \equiv \frac{1}{w_t} \phi^m_t$.

Let $[q(a, \bar{a}), p(a, \bar{a})]$ denote the terms at which a buyer who owns portfolio $a$ trades with a seller who owns portfolio $\bar{a}$, where $q(a, \bar{a}) \in \mathbb{R}^+$ is the quantity of special good traded, and $p(a, \bar{a}) = [p^s(a, \bar{a}), p^m(a, \bar{a})] \in \mathbb{R}^+ \times \mathbb{R}^+$ represents the transfer of assets from the buyer to the seller (the first argument is the transfer of equity). The value of search in the decentralized market satisfies

$$V(a, s) = \alpha \int \left\{ u[q(a, \bar{a})] + W[a - p(a, \bar{a}), s] \right\} dG(\bar{a})$$

$$+ \alpha \int \left\{ W[a + p(\bar{a}, a), s] - q(\bar{a}, a) \right\} dG(\bar{a}) + (1 - 2\alpha) W(a, s),$$

where $G$ denotes the distribution of portfolios.

Consider a meeting in the decentralized market between a buyer with portfolio $a_t$ and a seller with portfolio $\bar{a}_t$. The terms of trade, $(q_t, p_t)$, are determined by Nash bargaining where the buyer has all the bargaining power:

$$\max_{q_t, p_t \leq a_t} \left[ u(q_t) + W(a_t - p_t, s_t) - W(a_t, s_t) \right] \text{ s.t. } W(\bar{a}_t + p_t, s_t) - q_t \geq W(\bar{a}_t, s_t).$$
The constraint $p_t \leq a_t$ indicates that the buyer in a bilateral meeting cannot spend more than the assets he owns. Since $W(a_t + p_t, s_t) - W(a_t, s_t) = \lambda_t p_t$, the bargaining problem is

$$\max_{q_t, p_t \leq a_t} [u(q_t) - \lambda_t p_t] \quad \text{s.t.} \quad \lambda_t p_t - q_t \geq 0.$$ 

If $\lambda_t a_t \geq q^*$, the buyer gets $q_t = q^*$ special goods in exchange for a vector $p_t$ of assets with real value $\lambda_t p_t = q^* \leq \lambda_t a_t$. Else, the buyer gives the seller $p_t = a_t$, in exchange for $q_t = \lambda_t a_t$ special goods. Hence, the quantity of output exchanged is $q(\lambda_t a_t)$, with $q(\cdot)$ given by (3). Note that, if $\lambda_t a_t < q^*$, and $\lambda_t a_t \geq q^*$, and $\partial q(\lambda_t a_t) \partial a_t^2 = 0$. With the bargaining solution, the value function in the decentralized market becomes

$$V(a_t, s_t) = S(\lambda_t a_t) + W(a_t, s_t), \quad (21)$$

with $S(\cdot)$ given by (5).

The agent’s problem in the second subperiod is summarized by (20). The necessary and sufficient first-order conditions for the choices of $a_{t+1}^* \text{ and } a_{t+1}^m$ are

$$U'(c_t) \phi_t^s = \beta E_t V_{a_{t+1}}(a_t, s_{t+1})$$

$$U'(c_t) \phi_t^m \geq \beta E_t V_{a_{t+1}}(a_t, s_{t+1}), \quad \text{if } a_{t+1}^m > 0.$$ 

Again, the agent’s choices do not depend on his individual asset holdings, and the distribution of assets, $G$, will be degenerate (at the mean) in equilibrium. From (21), $1/\lambda_t^2 = \frac{1}{\lambda_t} \frac{\partial V(a_t, s_t)}{\partial a_t} = 1 - \alpha + \alpha u'(q(\lambda_t a_t))$, so these first-order conditions can be written as

$$U'(c_t) \phi_t^s = \beta \int \{1 - \alpha + \alpha u'(q(\lambda_{t+1} a_{t+1}))\} \lambda_{t+1}^s dF(d_{t+1}, d_t)$$

$$U'(c_t) \phi_t^m \geq \beta \int \{1 - \alpha + \alpha u'(q(\lambda_{t+1} a_{t+1}))\} \lambda_{t+1}^m dF(d_{t+1}, d_t), \quad \text{if } m_{t+1} > 0.$$ 

Let $\mu : \Xi \rightarrow \mathbb{R}^+$, and suppose that the government follows a stationary monetary policy, $M_{t+1} = \mu(d_t) M_t$. For the positive analysis I will focus on admissible monetary policies, i.e., $\mu \in C^+$, where the space of continuous, bounded, and nonnegative real-valued functions 

\[\text{Note that } \frac{\partial^2 V(a_t, s_t)}{\partial(a_t)^2} = \alpha u''(q(\lambda_t a_t)) (\lambda_t^2) \text{ if } \lambda_t a_t < q^* \text{, and } \frac{\partial^2 V(a_t, s_t)}{\partial(a_t)^2} = 0 \text{ if } \lambda_t a_t \geq q^*. \text{ So } \frac{\partial^2 V(a_t, s_t)}{\partial(a_t)^2} \leq 0 \text{ (and } < 0 \text{ if } \lambda_t a_t < q^*). \text{ Also, } \frac{\partial^2 V(a_t, s_t)}{\partial(a_t^2) \partial\lambda_t} = \alpha u''(q(\lambda_t a_t)) (\lambda_t^2) \text{, and } \frac{\partial^2 V(a_t, s_t)}{\partial(a_t^2) \partial\lambda_t} = \frac{\partial^2 V(a_t, s_t)}{\partial(a_t^2) \partial\lambda_t} \text{, so } V(a_t, d_t) \text{ is concave in } a_t \text{ (strictly concave for } \lambda_t a_t < q^*).} \]
on $\Xi$. Let $s_t = (x_t, M_t)$ denote the state of the economy at time $t$. The transition function $F$ together with the policy function $\mu$, induces a transition function for $s_t$, i.e., if $s = (x, M)$, and $s' = (x', M')$, then $Pr(s_{t+1} \leq s' | s_t = s) = I_{(\mu(x), M \leq M')} F (x', x) \equiv F (s', s)$. Let $\Psi$ be the associated stationary distribution, i.e., let $\Psi$ be the unique solution to $\Psi (s') = \int F (s', s) d\Psi (s)$.

I will consider a recursive equilibrium in which all prices are time-invariant functions of the agent’s problem in the centralized market; (ii) the terms of trade are determined by Nash bargaining, i.e., given prices and the bargaining protocol, the decision rules are such that the centralized market clears, i.e., $a > \lambda (s_t) = [\lambda^s (s_t), \lambda^m (s_t)]$, where $\lambda^s (s_t) \equiv \frac{1}{w(s_t)} [\phi^s (s_t) + x_t]$ and $\lambda^m (s_t) = \frac{1}{w(s_t)} \phi^m (s_t)$.

**Definition 2** Given a monetary policy rule $\mu$, a recursive equilibrium is a collection of individual decision rules $c_t = c(s_t)$, $a_{t+1} = a(s_t) = [a^s (s_t), a^m (s_t)]$, pricing functions $w_t = w(s_t)$, $\phi^s_t = \phi^s (s_t)$, $\phi^m_t = \phi^m (s_t)$, and bilateral terms of trade $q_t = q(s_t)$ and $p_t = p(s_t)$ such that: (i) given prices and the bargaining protocol, the decision rules $c(\cdot)$, and $a(\cdot)$, solve the agent’s problem in the centralized market; (ii) the terms of trade are determined by Nash bargaining, i.e., $q(s_t) = q (\lambda (s_t) a (s_t))$ and $\lambda (s_t) p (s_t) = \min [\lambda (s_t) a (s_t), q]$; and (iii) prices are such that the centralized market clears, i.e., $c(s_t) = d_t$, $a^s (s_t) = 1$. The equilibrium is monetary if $\phi^m (s_t) > 0$ for all $s_t$, and in this case the money-market clearing condition is $a^m (s_t) = \mu (d_t) M_t$.

From (6), if the current state is $s = (x, M)$, then

$$
\lambda^s (s) = U' (x) [\phi^s (s) + x] \quad \text{and} \quad \lambda^m (s) = U' (x) \phi^m (s).
$$

In words, $\lambda^s (s)$ is the real value (in terms of general goods) of an agent’s equilibrium equity holding in the search market, and $\lambda^m (s)$ is the real value of a unit of money (also in terms of marginal utility of fruit). Let $z (s)$ represent the value of the equilibrium money holdings in state $s = (x, M)$, i.e.,

$$
z (s) \equiv \lambda^m (s) M, \quad (22)
$$

and let $\Lambda (s)$ be the value (in terms of general goods) of the equilibrium portfolio that an agent carries into the search market in state $s$, i.e.,

$$
\Lambda (s) \equiv \lambda^s (s) + z (s).
$$

In equilibrium, the Euler equations for equity and money holdings imply

$$
\lambda^s (s_t) = \beta \int L [\Lambda (s_{t+1})] \lambda^s (s_{t+1}) dF (s_{t+1}, s_t) + x_t U' (x_t) \quad (23)
$$

$$
z (s_t) \geq \frac{\beta}{\mu (x_t)} \int L [\Lambda (s_{t+1})] z (s_{t+1}) dF (s_{t+1}, s_t) \quad (24)
$$
respectively, where (24) holds with “=” if \( a^m (s_t) > 0 \), and
\[
L [\Lambda (s_{t+1})] \equiv 1 - \alpha + \alpha u' (\min \{ \Lambda (s_{t+1}), q^* \}) .
\]
Note that \( L (s_{t+1}) \geq 1 \) for all dividend realizations, \( x_{t+1} \in \Xi \), with \( L (s_{t+1}) > 1 \) for \( x_{t+1} \in \Omega^m (s_t) \), where
\[
\Omega^m (s_t) = \{ x_{t+1} \in \Xi : \lambda^s [x_{t+1}, \mu (x_t) M_t] + \lambda^m [x_{t+1}, \mu (x_t) M_t] \mu (x_t) M_t < q^* \} .
\]
Notice that \( z (s_t) = 0 \) for all \( s_t \) solves (24), and also, that \( z (s_t) = 0 \) for all \( s_t \) implies \( \Lambda (s_t) = \lambda^s (s_t) \) for all \( s_t \), so a nonmonetary equilibrium exists provided the functional equation (9) has a solution.

4.1 Observables

In this section I derive expressions for the nominal interest rate, the rate of increase in nominal prices, the return on equity, and other empirically observable functions of the equilibrium allocations and prices. In Section 6, I will discuss the relationships between these variables to highlight some positive predictions of the theory.

Relative prices. There are three goods in this economy: special goods, general goods, and fruit. Recall that \( \phi^m (s) \), \( \phi^s (s) \), and \( w (x) \) (which equals \( 1/U' (x) \)) are expressed in terms of fruit. The bargaining solution implies that in every bilateral trade, the buyer hands over a portfolio of assets that is worth \( \min (\Lambda (s), q^*) \) general goods, in exchange for \( \min (\Lambda (s), q^*) \) special goods. Hence, the relative price of special goods in terms of general goods equals 1.

The following table summarizes the full set of relative prices.

<table>
<thead>
<tr>
<th>Price of</th>
<th>in terms of</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td>special good</td>
</tr>
<tr>
<td>special good</td>
<td>1</td>
</tr>
<tr>
<td>general good</td>
<td>1</td>
</tr>
<tr>
<td>fruit</td>
<td>( U'(x) )</td>
</tr>
<tr>
<td>money</td>
<td>( U'(x) \phi^m(s) )</td>
</tr>
</tbody>
</table>

Table 1: Relative prices in state \( s = (x, M) \)

Nominal interest rate. In order to derive an expression for the “shadow” nominal interest rate, imagine there existed an additional asset in this economy, a one-period risk-free bond
that pays a unit of money in the centralized market, which cannot be used in the decentralized exchange. Let \( \phi^n (s_t) \) denote the state-s_t price of this nominal bond. In equilibrium, this price must satisfy
\[
U' (x_t) \phi^n (s_t) = \beta \int U' (x_{t+1}) \phi^m (s_{t+1}) dF (s_{t+1}, s_t).
\] (25)

Notice that \( \frac{\phi^n (s_t)}{\phi^m (s_t)} \) is the dollar price of a nominal bond in state \( s_t \), so \( 1 + i (s_t) = \frac{\phi^m (s_t)}{\phi^n (s_t)} \) is the gross nominal interest rate in state \( s_t \). Hence, in a monetary equilibrium,
\[
1 + i (s_t) = \frac{\int L [\Lambda (s_{t+1})] z (s_{t+1}) dF (s_{t+1}, s_t)}{\int z (s_{t+1}) dF (s_{t+1}, s_t)}.
\] (26)

**Inflation.** The price of money, \( \phi^m (s) \), is quoted in terms of fruit. Since the relative price of fruit in terms of general goods in state \( s = (x, M) \) is \( U' (x) \), the price of money in terms of general goods is \( U' (x) \phi^m (s) = \lambda^m (s) \). Let
\[
\pi_f (s', s) = \frac{\phi^m (s)}{\phi^m (s')} - 1
\] (27)
denote the change in the dollar price of fruit between state \( s = (x, M) \) and a next-period state \( s' = (x', \mu (x) \ M) \) that follows from \( s \) under a monetary policy \( \mu \). Similarly, let
\[
\pi_g (s', s) = \frac{\lambda^m (s)}{\lambda^m (s')} - 1
\] (28)
denote the change in the dollar price of the general good. Expected (gross) inflation as measured by the dollar price of the general good, conditional on the information available in state \( s = (x, M) \), under the monetary policy \( \mu \), is \( 1 + \tilde{\pi}_g (s) = \int [1 + \pi_g (s', s)] dF (s', s) \), where \( s' = (x', \mu (x) \ M) \). Expected inflation measured by the dollar price of fruit, \( \tilde{\pi}_f (x) \), is defined analogously. The average (long-run) inflation rate, measured by the dollar price of good \( i = f, g \), is
\[
\bar{\pi}_i = \int \tilde{\pi}_i (s) d\Psi (s).
\]

**Real interest rate.** In order to derive an expression for the real interest rate, imagine there existed an additional asset in this economy, a one-period risk-free bond that pays a unit of fruit in the centralized market, which cannot be used in the decentralized exchange. Let \( \phi^r (\cdot) \) denote the price function corresponding to this real bond. In equilibrium, it must satisfy
\[
U' (x_t) \phi^r (x_t) = \beta \int U' (x_{t+1}) dF (x_{t+1}, x_t).
\] (29)
Notice that $\phi^r(x_t)$ is the relative price of a sure claim to fruit in period $t+1$ in terms of fruit in state $s_t = (x_t, M_t)$, so

$$1 + r(x_t) = \frac{1}{\phi^r(x_t)}$$

is the gross real interest rate in state $s_t$. In a monetary equilibrium, we can combine (26) with (22), (24), (29), and (30) to arrive at

$$\frac{1 + r(x_t)}{1 + i(s_t)} = \omega(x_{t+1}, x_t) [1 + \pi_f(s_{t+1}, s_t)]^{-1} dF(s_{t+1}, s_t),$$

where $\omega(x_{t+1}, x_t) \equiv U_0(x_{t+1}) - 1$. Condition (31) is a stochastic generalization of the simple Fisher equation: it equates the real-to-nominal gross interest rate ratio, to the conditional expectation of the marginal-utility-of-fruit-weighted reciprocal of the inflation rate (in this case, as measured by the price of fruit).9

Output. In state $s = (x, M)$, the quantity of fruit equals the endowment, $x$. The quantity of special goods produced equals $\alpha \min(\Lambda(s), q^*)$. Production of general goods is carried out by agents who acted as buyers in the previous round of decentralized trade. From the budget constraint that each agent faces in the centralized market, we see that in order to replenish his asset holdings, each agent who was a buyer in the previous decentralized market needs to produce $\min(\Lambda(s), q^*)$ general goods. Hence the total output of general goods in state $s$ is $\alpha \min(\Lambda(s), q^*)$. Aggregate output, expressed in terms of general (or special) goods, is $Y_g(s) = xU^r(x) + 2\alpha \min(\Lambda(s), q^*)$. Aggregate output expressed in terms of fruit is $Y_f(s) = Y_g(s) / \phi^m(s)$. Nominal aggregate output in state $s = (x, M)$ is $Y_n(s) = Y_f(s) / \phi^m(s)$.

Real return on equity. The real (in terms of fruit) gross return on equity between state $s = (x, M)$ and a next-period state $s' = (x', M')$ is $R^s(s', s) = \frac{\phi^m(s') + x'}{\phi^m(s)}$. The expected return on equity, conditional on the information available in state $s$, under monetary policy $\mu$ is $\tilde{R}^s(s) = \int R^s(s', s) dF(s', s)$. The average (long-run) equity return is $\bar{R}^s = \int \tilde{R}^s(s) d\Psi(s)$.

5 Optimal monetary policy

The Pareto optimal allocation can be found by solving the problem of a planner who wishes to maximize average (equally-weighted) expected utility. Given the initial condition $d_0 \in \Xi$, the

---

9Svensson (1985) derives an expression like (31) for his cash-in-advance economy.
planner’s problem consists of finding the sequence \( \{c_t, q_t\}_{t=0}^{\infty} \) that achieves

\[
V^* = \max_{\{c_t, q_t\}_{t=0}^{\infty}} \mathcal{W}(\{c_t, q_t\}_{t=0}^{\infty})
\]

\[
\text{s.t. } 0 \leq q_t, 0 \leq c_t \leq d_t,
\]

where

\[
\mathcal{W}(\{c_t, q_t\}_{t=0}^{\infty}) = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \{ \alpha [u(q_t) - q_t] + U(c_t) \} \right\}.
\]

The conditional expectation \( E_0 \) is defined with respect to the transition probability \( F(d_{t+1}, d_t) \).

The solution is to set \( \{c_t, q_t\}_{t=0}^{\infty} = \{d_t, q^*\}_{t=0}^{\infty} \). From Definition 2, it is clear that the competitive allocation in the centralized market always coincides with the efficient allocation. However, the equilibrium allocation may have \( q_t < q^* \) in some states. That is, in general, consumption and production in the decentralized market may be too low in a monetary equilibrium.

**Proposition 1** Let \( k \) be an arbitrary constant with \( k \geq 1 \), and \( \tilde{z} : \Omega \rightarrow \mathbb{R}^+ \) be an arbitrary bounded function. Define

\[
z^* (x) = \left\{ \begin{array}{ll} kq^* - \lambda^* (x) & \text{if } x \in \Omega, \\ \tilde{z}(x) & \text{if } x \in \Omega^c, \end{array} \right.
\]

where \( \lambda^* (x) \) is the unique continuous, bounded, and strictly positive solution to (15).

(i) There exists a recursive monetary equilibrium under the monetary policy

\[
\mu^* (x) = \beta \int z^* (x') dF(x', x) z^* (x),
\]

and the equilibrium prices of equity and money are

\[
\phi^e (x) = \frac{\lambda^* (x) - xU' (x)}{U' (x)}
\]

and

\[
\phi^m (s) = \frac{z^* (x)}{U' (x) M}
\]

for \( x \in \Xi, \) and \( s = (x, M) \in \Xi \times \mathbb{R}^+ \).

(ii) The monetary policy \( \mu^* \) is optimal.

(iii) The nominal interest rate is constant and equal to 0 under the policy (34).

(iv) The equilibrium state-by-state gross inflation rate is

\[
1 + \pi^*_f (x', x) = \beta \int z^* (x') dF(x', x) U' (x') z^* (x') U' (x)
\]
if measured by the price of fruit, or

\[ 1 + \pi_g^* (x', x) = \beta \int \frac{z^*(x') \, dF(x', x)}{z^*(x')} \]  

(38)

if measured by the price of general goods. Let \( \bar{\pi}_g^* \) denote the average (long-run) inflation in the price of general goods. Then, \( \bar{\pi}_g^* \geq \beta - 1 \), with strict inequality unless \( z^* \) is a degenerate random variable.

Part (i) of Proposition 1 characterizes a family of optimal stochastic monetary policies indexed by the number \( k \) and the function \( \bar{z} \). A monetary policy in this class induces \( q_t = q^* \) in every bilateral trade for every realization of the aggregate state. Under (34), the pricing function (35) is identical to the one derived for the nonmonetary economy with \( \Omega = \emptyset \), which is the pricing function in Lucas (1978). An optimal monetary policy ensures that the marginal return to the agent from carrying an additional dollar into the decentralized market is zero, i.e., equal to the government’s (marginal) cost of providing real balances. An optimal policy induces agents to hold just enough money so that \( L[\Lambda(s)] = 1 \) with probability 1 in the equilibrium. Part (iii) confirms that a monetary policy in the family characterized in part (i) implements the Friedman rule: it induces a monetary equilibrium with the nominal interest rate equal to zero in every state.\(^{10}\) In deterministic environments, the Friedman rule can often be described by the simple deterministic monetary policy rule of deflating at the rate of time preference (see, e.g., Lagos and Wright, 2005). In this environment there are stochastic liquidity needs, and the family of optimal policies in (34), is stochastic.\(^{11}\) According to part (iv), neither (37) nor (38) need to equal \( \beta \), and in fact, the long-run average inflation rate, \( \bar{\pi}_g^* \), exceeds \( \beta - 1 \) in general.

One way to think of the family defined in (34), is that it is constructed so that the real money balances \( z^* \) as defined in (33) can be part of a monetary recursive equilibrium. Since the constant \( k \geq 1 \) and the bounded function \( \bar{z} \) that define the real balances \( z^* \) are arbitrary, it is apparent that there is a large class of monetary policies for which there exists a monetary recursive equilibrium with zero nominal rate in every state. For example, Proposition 1 goes through if we replace \( z^* \) in (33) with any strictly positive, bounded function \( \hat{z} \) with the property

\(^{10}\)Even with no illiquid nominal bonds of the kind used to derive the nominal interest rate, (26), this marginal cost of holding nominal money balances (which in equilibrium equals the marginal benefit of holding money) has a natural interpretation as the price that the agent is willing to pay to own a dollar at the beginning of period \( t \) (and be able to use it in exchange) rather than at the end of period \( t \), after the round of decentralized trade.

\(^{11}\)In Lagos (2008) I characterize a large family of deterministic monetary policies that implement the Friedman rule in a generalization of this stochastic environment.
that $\dot{z}(x) + \lambda^s(x) \geq q^*$ for all $x \in \Xi$. The class of policies described by $z^*$ is of interest because as shown below (Proposition 3), it can be obtained as the limit of the class of policies that implement a constant (but possibly nonzero) nominal interest rate, as this target nominal interest rate approaches zero. Proposition 2, to which I turn next, provides another reason why the class of policies described in Proposition 1 is of interest.

Define the allocation rule $Q(s,\mu) : \Xi \times \mathbb{R}^+ \times C^+ \rightarrow \mathbb{R}^+$, and a price rule $\Phi^m(s,\mu) : \Xi \times \mathbb{R}^+ \times C^+ \rightarrow \mathbb{R}^+$. The allocation rule $Q(s,\mu)$ specifies the quantity of special goods traded in every bilateral meeting of a monetary recursive equilibrium under the policy rule $\mu$, in a period when the aggregate state is $s$. The price rule $\Phi^m(s,\mu)$ specifies the value of money (in terms of fruit) in a monetary recursive equilibrium under the policy rule $\mu$, in a period when the aggregate state is $s$.

**Proposition 2** Assume $B \leq (1 - \beta)q^*$, and let $\lambda^s(x)$ be the unique continuous, bounded, and strictly positive solution to (15). Let $\mu^*$ be as in (34), but with $z^*(x) = q^* - \lambda^s(x)$ for all $x \in \Xi$, and let

$$\mathcal{M} = \{ \mu \in C^+ : Q(s,\mu) = q^* \text{ for all } s \in \Xi \times \mathbb{R}^+ \}.$$

Then $\mu^* \in \mathcal{M}$, and for all $\mu \in \mathcal{M}$, $\Phi^m(s,\mu^*) \leq \Phi^m(s,\mu)$ for all $s \in \Xi \times \mathbb{R}^+$.

Proposition 2 corresponds to a parametrization for which, if the equity was priced as in an economy with no liquidity needs, agents would in fact experience a shortage of liquidity in every state. More formally, the assumption $B \leq (1 - \beta)q^*$ implies $\lambda^s(x) \leq q^*$ for all $x \in \Xi$, or equivalently, $\Omega = \Xi$. The proposition identifies, among the whole class of optimal monetary policies, the monetary policy that minimizes the value of money, and shows that this policy lies within the particular class of optimal policies described in Proposition 1.

**6 Asset prices, liquidity, and monetary policy**

In this section I consider a class of (possibly non-optimal) policies to study the relationship between asset prices, liquidity returns, and monetary policy from a positive standpoint. The class of optimal policies described in Proposition 1 can be characterized by the fact that they induce a nominal interest rate that is: (a) constant, and (b) equal to zero. The following proposition studies equilibrium under policies that induce a constant nominal interest rate, which is possibly higher than zero.
Proposition 3 Let $L(\delta) = 1 - \alpha + \alpha u(\delta q^*)$, and $\delta$ be defined by $L(\delta) = 1/\beta$. Let $\delta_0 \in (\delta, 1)$ be given, and suppose that $B \leq \left(1 - \beta L(\delta_0)\right) \delta_0 q^*$. For any $\delta \in [\delta_0, 1]$ define

$$z(x; \delta) = \delta q^* - \lambda(x; \delta)$$

(39)

where $\lambda(x; \delta)$ is the unique continuous, bounded, and strictly positive solution to (15), but with $\beta$ replaced by $\beta L(\delta)$.

(i) There exists a recursive monetary equilibrium under the monetary policy

$$\mu(x; \delta) = \beta L(\delta) \frac{\int z(x'; \delta) \, dF(x', x)}{z(x; \delta)},$$

(40)

and the equilibrium prices of equity and money are

$$\phi^s(x; \delta) = \frac{\lambda(x; \delta) - xU'(x)}{U'(x)}$$

(41)

$$\phi^m(s; \delta) = \frac{z(x; \delta)}{U'(x) M}$$

(42)

for $s = (x, M) \in \Xi \times \mathbb{R}^+$. 

(ii) The gross nominal interest rate in state $s$, $1 + i(s; \delta)$, is constant and equal to $L(\delta)$ under the policy (40).

(iii) The equilibrium state-by-state gross inflation rate is

$$1 + \pi_f(x', x; \delta) = \beta L(\delta) \frac{\int z(x'; \delta) \, dF(x', x) U'(x')}{z(x'; \delta)}$$

(43)

if measured by the price of fruit, or

$$1 + \pi_g(x', x; \delta) = \beta L(\delta) \frac{\int z(x'; \delta) \, dF(x', x)}{z(x'; \delta)}$$

(44)

if measured by the price of general goods.

(iv) Consider the recursive monetary equilibrium induced by the monetary policy (40) with $\delta \in (\delta_0, 1]$, and the recursive monetary equilibrium induced by (40) with $\delta' \in [\delta_0, \delta)$. Then: (a) $\lambda(\cdot; \delta') < \lambda(\cdot; \delta')$, (b) $z(x; \delta') < z(x; \delta)$, (c) $\phi^s(\cdot; \delta') < \phi^s(\cdot; \delta')$, (d) $\phi^m(\cdot; \delta') < \phi^m(\cdot; \delta)$, (e) $i(\cdot; \delta') < i(\cdot; \delta')$. Let $\bar{\pi}_g(x; \delta) \equiv \int \pi_g(x', x; \delta) \, dF(x', x)$, then: (f) $1 + \bar{\pi}_g(x; \delta) \geq \beta L(\delta)$ (with strict inequality unless $\lambda(x; \delta)$ is a degenerate random variable).

(v) $\lim_{\delta \to 1} \mu(x; \delta) = \mu(x; 1) = \mu^*(x)$, with $\mu^*(x)$ as given in Proposition 2.
The class of monetary policies described in part (i) is indexed by the parameter $\delta$, which according to part (ii), effectively determines the level of the constant nominal interest rate that the policy targets. Part (iv) shows that under the proposed policy, the price of equity is increasing in the nominal interest rate target (decreasing in $\delta$), while real balances and the value of money are decreasing in the nominal interest rate implemented by the policy (increasing in $\delta$). Expected inflation as measured by the dollar price of general goods, conditional on the information available in state $s = (x, M)$, and $\bar{\pi}_g(\delta) \equiv \int \bar{\pi}_g(x; \delta) d\psi(x)$, are bounded below by $\beta_L(\delta)$. Part (v) shows that as $\delta \to 1$ the policy $\mu(x; \delta)$ approaches the optimal policy described in Proposition 2, and therefore the monetary equilibrium characterized by (39)–(42) converges to the efficient equilibrium of Proposition 2.

Proposition 3 provides insights on how the monetary policy (40) can support a recursive monetary equilibrium with a constant nominal interest rate, with the optimal equilibrium in which the nominal rate is constant and zero as a special case. According to (40), the money growth rate should be relatively low in states in which the real value of the equilibrium equity holdings, is below average. For example, with $\delta = 1$ (the optimal policy of Proposition 2), $\mu^*(x) < \beta$ if and only if $\lambda(x) < \int x'(x') dF(x', x)$, and $\mu^*(x) = \beta$ if $\lambda(x) = \int x'(x') dF(x', x)$. Something similar happens with the implied inflation rate. From (44), for example, the inflation rate between state $x$ and a next-period state $x'$ is relatively low if the realized real value of the equilibrium equity holdings in state $x'$ is below the conditional expectation held in state $x$.

**Corollary 1** Consider the economy described in Proposition 3, with $dF(x', x) = dF(x)$. Then for any $\delta \in [\delta_0, 1]$, there exists a recursive monetary equilibrium under the monetary policy

$$\mu(x; \delta) = \beta L(\delta) \frac{\delta q^* - \frac{1}{1 - \beta L(\delta)} \int x' U'(x') dF(x')}{\delta q^* - \frac{\beta L(\delta)}{1 - \beta L(\delta)} \int x' U'(x') dF(x') - x U'(x)},$$

and the equilibrium prices of equity and money are

$$\phi^s(x; \delta) = \frac{\beta L(\delta) \int x' U'(x') dF(x')}{1 - \beta L(\delta) U'(x)}$$

$$\phi^m(s; \delta) = \frac{\delta q^* - \frac{\beta L(\delta)}{1 - \beta L(\delta)} \int x' U'(x') dF(x') - x U'(x)}{U'(x) M}$$

for $s = (x, M) \in \Xi \times \mathbb{R}^+$. The state-by-state inflation rate measured by the price of fruit is

$$\pi_f(x', x; \delta) = \frac{U'(x')}{U'(x)} \mu(x'; \delta) - 1.$$
The state-by-state inflation rate measured by the price of general goods is

\[ \pi_g(x', x; \delta) = \mu(x'; \delta) - 1. \]  

As long as \( B \leq (1 - \beta) q^* \), the monetary policy and the equilibrium obtained in Corollary 1 with \( \delta = 1 \), coincide with the optimal monetary policy and the efficient equilibrium we would obtain if we assumed \( dF(x', x) = dF(x) \) in Proposition 2. From (45), for every \( \delta \in [\delta_0, 1] \), \( \partial\mu(x; \delta) / \partial x \) is proportional to \( 1 - \rho(x) \), i.e., the rate of money creation is procyclical if \( \rho(x) < 1 \). In particular, notice that this is also true for the optimal policy, \( \mu(x; 1) \). From (46), the level of asset prices is increasing in the nominal interest rate, \( L(\delta) \), so the state-by-state real return on equity,

\[ R^s(x', x; \delta) = \left[ 1 + \frac{1 - \beta L(\delta) U'(x')}{\beta L(\delta)} x' \int x' U'(x') dF(x') \right] \frac{U'(x)}{U'(x')}, \]  

is decreasing in the nominal interest rate. A higher nominal interest rate implies that buyers are on average short of liquidity, so equity becomes more valuable as it is used by buyers to relax their trading constraints. This additional liquidity value means that the real financial return on equity, e.g., (50), will be lower, on average, at a higher interest rate.

**Corollary 2** Consider the economy described in Proposition 3, with \( U(c) = \log c \), and assume \( 1 < [1 - \beta L(\delta_0)] \delta_0 q^* \). For any \( \delta \in [\delta_0, 1] \), there exists a recursive monetary equilibrium under the monetary policy

\[ \mu(x; \delta) = \beta L(\delta), \]  

and the equilibrium prices of equity and money are

\[ \phi^s(x; \delta) = \frac{\beta L(\delta)}{1 - \beta L(\delta)} x, \]  

\[ \phi^m(s; \delta) = \frac{\delta q^* - 1 - \beta L(\delta)}{M} x \]  

for \( x \in \Xi \), and \( s = (x, M) \in \Xi \times \mathbb{R}^+ \). The gross nominal interest rate in state \( s \), i.e., \( 1 + i(s; \delta) \), is constant and equal to \( L(\delta) \) under the policy (51). The equilibrium gross state-by-state gross inflation rate is

\[ 1 + \pi_f(x', x; \delta) = \beta L(\delta) \frac{x}{x'} \]  

if measured by the price of fruit, or

\[ 1 + \pi_g(x', x; \delta) = \beta L(\delta). \]
As long as $B \leq (1 - \beta) q^*$, the monetary policy and the equilibrium obtained in Corollary 1 with $\delta = 1$, coincide with the optimal monetary policy and the efficient equilibrium we would obtain if we assumed $U(c) = \log c$ in Proposition 2.

7 Conclusion

In this paper I have formulated a model in which agents hold assets not only for their intrinsic value, but also to use them as means of payment. In particular, I have considered a financial structure with two assets, equity shares of a stochastic real exogenous dividend, and fiat money. In this context, I characterized a family of optimal, stochastic monetary policy rules. Every policy in this family implements Friedman’s prescription of zero nominal interest rates. Under an optimal policy, equity prices and returns are independent of monetary considerations.

I have also studied a class monetary policies that target a constant, but nonzero nominal interest rate. For this perturbation of the family of optimal policies, I have found that the model articulates the idea that, to the extent that a financial asset may be used to facilitate transactions, this liquidity service will be priced in the asset and reflected in its measured financial return. In addition, even if the asset is real, as would be the case with an equity share that represents a claim to a real exogenous dividend stream, whenever it serves this liquidity role alongside a monetary asset, the equity price and rate of return will depend on monetary considerations. Since agents are free to use any combination of assets for exchange purposes, even if the equity is real and its dividend exogenous, part of its return will be linked to its liquidity return, and this liquidity return in turn depends on the quantity of real money balances—which is a function of the inflation rate. On average, if the rate of inflation is higher, real money balances are lower, and the liquidity return on equity rises, which causes its price to rise and its real measured rate of return (dividend yield plus capital gains) to fall. This type of logic could help to rationalize the fact that historically, real stock returns and inflation have been negatively correlated—an observation long considered anomalous in the finance literature.\footnote{See, e.g., Bodie (1976), Bordo et al (2008), Boudoukh and Richardson (1993), Fama (1981), Fama and Schwert (1977), Gultekin (1983), Jaffe and Mandelker (1976), Kaul (1987), Marshall (1992), Nelson (1976).}

The model has a number of implications for the time-paths of output, inflation, interest rates, equity prices, and equity returns, and it would be interesting to explore these implications further. For example, even though variations in aggregate output are effectively exogenous under the types of monetary policies that were considered, the theory can produce a negative
correlation between the inflation rate and the growth rate of output—a short-run “Phillips curve”—but one that is entirely generated by a monetary policy designed to target a constant nominal interest rate in an economy with stochastic liquidity needs.
A Proofs

Proof of Lemma 1. Let

\[ \Upsilon(\Delta) \equiv \int \{1 - \alpha + \alpha u' \left( \min \left\{ \beta \Delta + xU'(x), q^* \right\} \right) \} \left[ \beta \Delta + xU'(x) \right] dF(x) - \Delta, \]

then \( \Upsilon(\Delta) = 0 \) is equivalent to (10). Note that \( \Upsilon(0) > 0 \). Also, \( \Upsilon(\Delta) = \int xU'(x) dF(x) - (1 - \beta) \Delta \) for all \( \Delta \geq q^*/\beta \), so \( \lim_{\Delta \to -\infty} \Upsilon(\Delta) = -\infty \). Since \( \Upsilon \) is continuous, there exists a \( \Delta > 0 \) such that \( \Upsilon(\Delta) = 0 \). Differentiate \( \Upsilon(\Delta) \) to get

\[ \Upsilon'(\Delta) = - (1 - \beta) + \alpha \beta \int_\Omega \left[ \beta \Delta + xU'(x) \right] u'' dF(x) + \alpha \beta \int_\Omega (u' - 1) dF(x) \]

where \( \Omega = \{ x \in \Xi : \beta \Delta + xU'(x) < q^* \} \), and with \( u'' \) and, \( u' \) evaluated at \( \beta \Delta + xU'(x) \). Note that \( \Upsilon(\Delta) = 0 \) implies

\[ \alpha \beta \int_\Omega (u' - 1) dF(x) = 1 - \beta - \varsigma, \]

where \( \varsigma = \frac{1}{\Delta} \int (1 - \alpha + \alpha u') xU'(x) dF(x) > 0 \). Therefore

\[ \Upsilon'(\Delta)|_{\Upsilon(\Delta)=0} = \alpha \beta \int_\Omega u'' \lambda(z) F(z) - \varsigma < 0, \]

so \( \Upsilon(\Delta) = 0 \) has a unique solution. Finally,

\[ \frac{\partial \Upsilon(\Delta)}{\partial \alpha} = \int_\Omega (u' - 1) \left[ \beta \Delta + xU'(x) \right] dF(x) \geq 0, \]

“>” if \( \Omega \neq \emptyset \), so

\[ \frac{d\Delta}{d\alpha} \bigg|_{\Upsilon(\Delta)=0} = \frac{\partial \Upsilon(\Delta)/\partial \alpha}{-\Upsilon'(\Delta)|_{\Upsilon(\Delta)=0}} \geq 0, \]

“>” if \( \Omega \neq \emptyset \). ■

Proof of Claim 1. (i) Note that \( q^* - \beta \Delta \leq 0 \) if and only if \( \Upsilon(q^*/\beta) \geq 0 \), where the function \( \Upsilon \) is defined in (56). It is straightforward to verify that \( \Upsilon(q^*/\beta) \geq 0 \) if and only if \( q^* \leq \frac{1}{\beta} \int zU'(z) dF(z) \). But if this is the case, then \( \Omega = \emptyset \) since \( xU'(x) \geq 0 \) for all \( x \).

(ii) \( \rho(x) > 1 \) implies that \( xU'(x) \) is strictly decreasing. The condition in part (ii)(a) implies that \( xU'(x) \geq q^* - \Delta \) for all \( x \). The condition in part (ii)(c) implies that \( xU'(x) < q^* - \Delta \) for all \( x \). Since \( xU'(x) \) is continuous and strictly decreasing, the condition in part (ii)(b) implies there exists a unique \( x^* \in [x, \infty) \) characterized by \( x^*U'(x^*) = q^* - \beta \Delta \) such that \( xU'(x) < q^* - \Delta \) if and only if \( x > x^* \).

(iii) \( \rho(x) < 1 \) implies that \( xU'(x) \) is strictly increasing. The condition in part (iii)(a) implies that \( xU'(x) < q^* - \Delta \) for all \( x \). The condition in part (iii)(c) implies that \( xU'(x) \geq q^* - \Delta \) if and only if \( x < x^* \).
\[ q^* - \Delta \text{ for all } x. \] Since \( xu'(x) \) is continuous and strictly increasing, the condition in part (iii)(b) implies there exists a unique \( x^* \in (\xi, \infty) \) characterized by \( xu'(x^*) = q^* - \beta \Delta \), such that \( xu'(x) < q^* - \Delta \) if and only if \( x < x^* \).

(iv) \( \rho(x) = 1 \) implies \( \Omega = \{x \in \Xi : 1 + \beta \Delta < q^*\} \), so either \( \Omega = \emptyset \) or \( \Omega = \Xi \). Note that \( 1 + \beta \Delta < q^* \), and hence \( \Omega = \Xi \), if and only if \( \Upsilon((q^* - 1)/\beta) < 0 \). It is straightforward to verify that \( \Upsilon((q^* - 1)/\beta) < 0 \) if and only if \( q^* > \frac{1}{1 - \beta} \). Hence, \( q^* > \frac{1}{1 - \beta} \) implies \( \Omega = \Xi \), and \( q^* \leq \frac{1}{1 - \beta} \) implies \( \Omega = \emptyset \). \]  

**Proof of Lemma 2.** Let \( C \) denote the space of continuous and bounded real-valued functions defined on \( \mathbb{R}^+ \). The right side of (14) defines a mapping \( T \), i.e., for any \( g \in C \),

\[
(Tg)(x) = xu'(x) + \beta \int \{(1 - \alpha)g(x') + \alpha \max \{g(x'), 1\}\} dF(x', x).
\]

\( U \) is continuously differentiable, concave, and bounded, with \( U(0) = 0 \), so \( xu'(x) \in C \). Also, \( (1 - \alpha)g(x) + \alpha \max \{g(x), 1\} \in C \), and since \( F \) has the Feller property, \( Tg \in C \). Hence \( T : C \to C \). Notice that for all \( f, g \in C \) such that \( f(x) \leq g(x) \) for all \( x \in \mathbb{R}^+ \), \( (Tf)(x) \leq (Tg)(x) \) for all \( x \in \mathbb{R}^+ \). Let \( \Xi^+ = \{x \in \Xi : g(x) > 1\} \), and \( \Xi^- = \Xi \setminus \Xi^+ \). Then for all \( k, x \in \mathbb{R}^+ \),

\[
[T(g + k)](x) - (Tg)(x) = \beta \int \{(1 - \alpha)[g(x') + k] + \alpha \max \{g(x') + k, 1\}\} dF(x', x)
- \beta \int \{(1 - \alpha)g(x') + \alpha \max \{g(x'), 1\}\} dF(x', x)
= \beta (1 - \alpha)k + \beta \alpha \int \max \{g(x') + k, 1\} dF(x', x)
- \beta \alpha \int \max \{g(x'), 1\} dF(x', x)
= \beta (1 - \alpha)k + \beta \alpha \int_{\Xi^+} dF(x', x) k
+ \beta \alpha \int_{\Xi^-} \max \{g(x') - 1 + k, 0\} dF(x', x)
\leq \beta k.
\]

Hence, if we let \( \|\cdot\| \) denote the sup norm, i.e., \( \|f - g\| = \sup_{x \in \mathbb{R}^+} |f(x) - g(x)| \) for any \( f, g \in C \), \( T \) satisfies Blackwell’s sufficient conditions (Theorem 3.3 in Stokey and Lucas, 1989), so \( T \) is a contraction mapping on the complete metric space \( (C, \|\cdot\|) \). By the Contraction Mapping Theorem (Theorem 3.2 in Stokey and Lucas, 1989), there exists a unique \( \lambda \in C \) that satisfies \( \lambda = T\lambda \). To show that \( \lambda > 0 \), for any \( g \in C \), define the mapping \( T^+ \) as

\[
(T^+g)(x) = xu'(x) + \beta \int \{(1 - \alpha) \max \{g(x'), 0\} + \alpha \max \{g(x'), 1\}\} dF(x', x).
\]
Clearly, $T^+: \mathcal{C} \to \mathcal{C}$, and since $T^+$ is a contraction on $(\mathcal{C}, \|\cdot\|)$, there exists a unique $\lambda^+ \in \mathcal{C}$ such that $\lambda^+ = T^+ \lambda^+$. For any $g \in \mathcal{C}$, $T^+ g > 0$, so $T^+ \lambda^+ = \lambda^+ > 0$. But notice that for any $g \in \mathcal{C}$ with $g > 0$, $Tg = T^+g$. Therefore $T \lambda^+ = T^+ \lambda^+ = \lambda^+$, so $\lambda = \lambda^+ > 0$. ■

Proof of Claim 2. Guess $\Omega = \Xi$, and substitute $U(c) = \varepsilon \log c$ in (17) to arrive at (19). Then note that $\lambda(x) = \frac{\beta \alpha + \varepsilon}{1 - \beta (1 - \alpha)} < 1$ if and only if $\varepsilon < 1 - \beta$, which verifies the guess. Clearly, there cannot exist an equilibrium with $\Omega = \Xi$ if $\varepsilon \geq 1 - \beta$, so in this case the equilibrium must have $\Omega \subset \Xi$. ■

Proof of Proposition 1. The functional equation (15) is a special case of (14), so by Lemma 2, it has a unique solution $\lambda^s$ in the space of continuous and bounded functions, $\mathcal{C}$, and this solution is strictly positive. (i) To show that there exists a monetary equilibrium under (34), construct one as follows. Let $z^*$ and $\lambda^s$ be defined as in the statement of the proposition. Let $s = (x, M)$ denote a state, and let $\mu(x)$ be some monetary policy. Consider the price functions $\phi^s(x)$ in (35), $\phi^m(s)$ in (35), and $w(x) = 1/U'(x)$, together with the allocation functions $c(s) = x$, $a^s(s) = 1$, $a^m(s) = \mu(x)M$, and $q(s) = q(\Lambda(x))$, with $\Lambda(x) = U'(x)[\phi^s(x) + x] + z^*(x)$. The claim to be established is that these allocation and price functions constitute a monetary recursive equilibrium if $\mu = \mu^s$. The allocation functions $c(s), a^s(s)$, and $q(s)$ clearly satisfy the equilibrium conditions. Moreover, $z^*(x) > 0$ for all $x$, so $\mu^s(x) > 0$ for all $x$, which implies that $a^m(s) = \mu^s(x)M > 0$ for all $s$, so $a^m(s)$ is consistent with a monetary equilibrium.

Next, we show that the proposed price functions indeed support the proposed allocations as a monetary recursive equilibrium under the monetary policy $\mu^s$. If $\mu = \mu^s$, in the conjectured equilibrium,

$$
\Lambda(x) = \lambda^s(x) + \left\{ \mathbb{I}_{\{x \in \Omega\}} \left[ kq^* - \lambda^s(x) \right] + \mathbb{I}_{\{x \in \Omega^c\}} \tilde{z}(x) \right\} \geq q^*
$$

for all $x \in \Xi$, (57) with strict inequality for $x \in \Omega^c$ (since $\tilde{z}(x) > 0$), and strict inequality for $x \in \Omega$ only if $k > 1$. Next, verify that (35) and (36) satisfy the Euler equations (23) and (24) under the monetary policy specified in (34). From (57), $L[\Lambda(s)] = 1$ for all $s \in \Xi \times \mathbb{R}^+$, so the Euler equation (24) reduces to

$$
z(x) \geq \frac{\beta}{\mu(x)} \int z(x') dF(x', x).
$$

(88)

(This condition would have to hold with “=” if we want to support a monetary equilibrium.) Notice that $z(x) = z^*(x)$ (simply substitute (36) into the definition (22)), so (88) holds with equality given that $\mu(x) = \mu^s(x)$. The fact that under $\mu^s$, $L[\Lambda(s)] = 1$ for all $s \in \Xi \times \mathbb{R}^+$, also
implies that the Euler equation (35) reduces to (15), and \( \lambda^s \) is the solution to (15), so (35) is satisfied in a monetary equilibrium under (34). Since \( \lambda^s \) solves (15), we have \( \lambda^s (x) - xU' (x) = \beta \int \lambda^s (x') dF (x', x) \), and since \( \lambda^s (x) > 0 \) for all \( x \), \( \phi^s (x) > 0 \) for all \( x \). Also, since \( z^* (x) > 0 \) for all \( x \in \Xi \), \( \phi^m (s) > 0 \) for all \( s \in \Xi \times \mathbb{R}^+ \), so the equilibrium constructed is indeed monetary.

In Lagos (2008) (Proposition 1), I show that the following transversality conditions must be satisfied in any equilibrium

\[
\lim_{t \to \infty} E_0 \left\{ \beta^t \frac{1}{w_t} \phi^{a^s_t} a^s_{t+1} \right\} = 0 \quad (59)
\]
\[
\lim_{t \to \infty} E_0 \left\{ \beta^t \frac{1}{w_t} \phi^{m^m} m^m_{t+1} \right\} = 0. \quad (60)
\]

To conclude, notice that evaluated at the prices and allocations of the proposed equilibrium, the left side of (59) becomes

\[
\lim_{t \to \infty} E_0 \left\{ \beta^t U' (x_t) \phi^s (x_t) \right\},
\]

and the left side of (60) becomes

\[
\lim_{t \to \infty} E_0 \left\{ \beta^t U' (x_t) \phi^m (s_t) \mu^* (x_t) M_t \right\}. \quad (62)
\]

With (35), (61) can be written as \( \lim_{t \to \infty} E_0 \left\{ \beta^t \left[ \lambda^s (x) - xU' (x) \right] \right\} \), and since \( \lambda^s (x) - xU' (x) \) is bounded (see the proof of Lemma 2),

\[
\lim_{t \to \infty} E_0 \left\{ \beta^t U' (x_t) \phi^s (x_t) \right\} = 0.
\]

With (34) and (36), (62) can be written as \( \lim_{t \to \infty} E_0 \left\{ \beta^{t+1} \int z^* (x') dF (x', x_t) \right\} \), and since \( z^* (x) \) is bounded,

\[
\lim_{t \to \infty} E_0 \left\{ \beta^t U' (x_t) \phi^m (s_t) \mu^* (x_t) M_t \right\} = 0.
\]

Therefore, the proposed allocation and price functions constitute a monetary recursive equilibrium under the monetary policy (34).

(ii) The monetary equilibrium constructed in part (i) has \( c (s) = x \), and from (57), also \( Q(s) = q^* \) for all \( s = (x, M) \), so it implements the optimal allocation that solves (32).

(iii) Immediate from (26), and the fact that \( L [\Lambda (s)] = 1 \) for all \( s \in \Xi \times \mathbb{R}^+ \) in the proposed equilibrium.

(iv) Combine (36) and (34) with (27) to arrive at (37). Combine (36) and (34) with (28) to arrive at (38). Let \( \tilde{\pi}_g^* (x) = \int \pi_g^* (x') dF (x', x) \), and \( \tilde{\pi}_g^* = \int \tilde{\pi}_g^* (x) d\psi (x) \). With (38) and
Jensen’s Inequality,

$$1 + \tilde{\pi}_g^* (x) = \beta \int \frac{z^*(x')dF(x',x)}{z^*(x')}dF(x',x) \geq \beta,$$

for all $x$, with strict inequality unless $z^*(x)$ is a degenerate random variable. This implies $\tilde{\pi}_g^* = \int \tilde{\pi}_g^* (x) d\psi (x) \geq \beta - 1$. ■

**Proof of Proposition 2.** The functional equation (15) is a special case of (14), so by Lemma 2, it has a unique solution $\lambda^s$ in the space of continuous and bounded functions, $C$, and this solution is strictly positive. Let $C'$ denote the space of continuous real-valued functions bounded by $q^s$ in the sup norm, and let $C''$ be the space of continuous real-valued functions, $g$, that satisfy $\sup_{x \in \Xi} |g(x)| < q^s$. Let $T$ be the mapping defined by $\langle Tg \rangle (x) = \beta \int g(x')dF(x',x) + xU''(x)$, so that $\lambda^s$ satisfies $\lambda^s = T\lambda^s$. Suppose $g \in C'$, then

$$\langle (Tg) \rangle (x) = \beta \int g(x')dF(x',x) + xU''(x) \leq \beta \sup_{x \in \Xi} |g(x)| + \sup_{x \in \Xi} |xU''(x)| < q^s,$$

where the last inequality follows from the fact that $g \in C'$, and

$$\sup_{x \in \Xi} |xU''(x)| < \sup_{x \in \Xi} |U'(x)| \leq B,$$

together with the hypothesis $B \leq (1 - \beta) q^s$. Thus, $T(C') \subseteq C'' \subseteq C'$, and since $C'$ is a closed subset of $C$, it follows (Corollary 1 in Stokey and Lucas (1989), p. 52) that $\lambda^s \in C''$. That is, $0 < \lambda^s(x) < q^s$, and therefore $z^*(x) > 0$ for all $x \in \Xi = \Omega$.

The monetary equilibrium induced by $\mu^*$ given $z^*(x) = q^s - \lambda^s(x)$ for all $x \in \Xi$, is the same monetary equilibrium induced by $\mu^*$ in Proposition 1 given (33), but for the special case with $\Omega^c = \emptyset$ and $k = 1$. Hence, $Q(s, \mu^*) = q^s$ for all $s \in \Xi \times \mathbb{R}^+$, so $\mu^* \in \mathcal{M}$. Next, define the **allocation rule** $Z(s, \mu) : B \times \mathbb{R}^+ \times C^+ \to \mathbb{R}^+$, with the interpretation that $Z(s, \cdot)$ represents the value of the equilibrium money holdings in state $s = (x, M)$, as defined in (22). Define the price rule $\Phi^s(s, \mu)$, i.e., the equity price in a monetary recursive equilibrium under the policy rule $\mu$, in a period when the aggregate state is $s$. Consider some $\mu \in \mathcal{M}$, such that $\mu \neq \mu^*$. Since $\mu \in \mathcal{M}$, $\Phi^s(s, \mu) = \Phi^s(s, \mu^*) = \phi^s(x)$, with $\phi^s(x)$ as given in (35). Also, $\mu \in \mathcal{M}$ implies

$$U''(x) [\phi^s(x) + x] + Z(s, \mu^*) = q^s \leq U''(x) [\phi^s(x) + x] + Z(s, \mu)$$

for all $s \in \Xi \times \mathbb{R}^+$. Since $Z(s, \mu) \equiv U''(x) \Phi^m(s, \mu) M$, this inequality implies $\Phi^m(s, \mu^*) \leq \Phi^m(s, \mu)$ for all $s \in \Xi \times \mathbb{R}^+$. ■
Proof of Proposition 3. (i) For any given \( \delta \in (\hat{\delta}, 1] \), the function \( \lambda (x; \delta) \) is the solution to

\[
\lambda (x) = \beta L (\delta) \int \lambda (x') \, dF (x', x) + x U' (x), \tag{63}
\]

which is the same as (15), but with discount factor \( \beta L (\delta) \in [\beta, 1) \), instead of \( \beta \in (0, 1) \). This functional equation is a special case of (14), so by Lemma 2, for any given \( \delta \in (\hat{\delta}, 1] \), it has a unique solution \( \lambda (\cdot; \delta) \) in the space of continuous and bounded functions, \( C \), and this solution is strictly positive. Let \( C' \) denote the space of continuous real-valued functions bounded by \( \delta_0 q^* \) in the sup norm, and let \( C'' \) be the space of continuous real-valued functions, \( g \), that satisfy \( \sup_{x \in \Xi} |g (x)| < \delta_0 q^* \). Let \( T_\delta \) denote the mapping defined by \( (T_\delta g) (x) = \beta L (\delta) \int g (x') \, dF (x', x) + x U' (x) \), so that \( \lambda (\cdot; \delta) \) satisfies \( \lambda (\cdot; \delta) = T_\delta \lambda (\cdot; \delta) \). If \( g \in C' \),

\[
| (T_\delta g) (x) | = \left| \beta L (\delta_0) \int g (x') \, dF (x', x) + x U' (x) \right| \leq \beta L (\delta_0) \sup_{x \in \Xi} |g (x)| + \sup_{x \in \Xi} |x U' (x)| < \delta_0 q^*,
\]

where the last inequality follows from the fact that \( g \in C' \), and

\[
\sup_{x \in \Xi} |x U' (x)| < \sup_{x \in \Xi} |U (x)| \leq B,
\]

together with the hypothesis \( B \leq [1 - \beta L (\delta_0)] \delta_0 q^* \). Thus, \( T_{\delta_0} (C') \subseteq C'' \subseteq C' \), and since \( C' \) is a closed subset of \( C \), it follows (Corollary 1 in Stokey and Lucas (1989), p. 52) that \( \lambda (\cdot; \delta_0) \in C'' \).

Hence

\[
0 < \lambda (x; \delta_0) < \delta_0 q^* \tag{64}
\]

and therefore \( z (x; \delta_0) > 0 \) for all \( x \in \Xi \).

Next, we establish that if \( \lambda (\cdot; \delta) = T_\delta \lambda (\cdot; \delta) \), and \( \lambda (\cdot; \delta') = T_{\delta'} \lambda (\cdot; \delta') \), for \( \delta, \delta' \in [\delta_0, 1] \), then

\[
\delta' < \delta \Rightarrow \lambda (\cdot; \delta) < \lambda (\cdot; \delta'). \tag{65}
\]

Let \( h (x) \equiv \lambda (x; \delta') - \lambda (x; \delta) \), then

\[
h (x) = \dot{\beta} \int h (x') \, dF (x', x) + v (x) \tag{66}
\]

where \( \dot{\beta} = \beta L (\delta') \in (0, 1) \), and \( v (x) \equiv \beta [L (\delta') - L (\delta)] \int \lambda (x'; \delta) \, dF (x', x) \), with \( L (\delta') - L (\delta) > 0 \). Notice that \( v \in C \) and \( v (x) > 0 \) for all \( x \), since \( \lambda (\cdot; \delta) \in C \), and \( \lambda (x; \delta) > 0 \) for all \( x \) (the properties of \( \lambda (\cdot; \delta) \) follow from Lemma 2, since \( \lambda (\cdot; \delta) \) is the fixed point of (63)). Therefore, by Lemma 2, the fixed point of (66) is strictly positive, i.e., \( h (x) > 0 \) for all \( x \), which implies (65).
Combined with (64), (65) implies that for every $\delta \in [\delta_0, 1]$, $\lambda(x; \delta) \leq \lambda(x; \delta_0) < \delta_0 q^* \leq \delta q^*$, so $z(x; \delta) > 0$ for all $x \in \Xi$ and all $\delta \in [\delta_0, 1]$.

For a fixed $\delta \in [\delta_0, 1]$, the equilibrium is constructed in a similar manner as in the proof of Proposition 2. Let $z(x; \delta)$ as given in (39) be the value of the equilibrium money holdings (expressed in terms of marginal utility of fruit), then $\Lambda(s) = \lambda(x; \delta) + z(x; \delta) = \delta q^*$ for all $s = (x, M) \in \Xi \times \mathbb{R}^+$, which implies $L[\Lambda(s)] = L(\delta)$ for all $s$. With this, (23) becomes (63), and as stated in the proposition, $\lambda(s; \delta)$ is the unique fixed point. The definition $\lambda(x; \delta) \equiv U'(x) [\phi^s(x) + x]$ then gives the equilibrium price function for equity, (41). Under the proposed monetary policy, (40), the Euler equation (24), holds with equality. Finally, (42) is obtained from the definition $z(x; \delta) \equiv U'(x) \phi^m(s) M$. The equilibrium is monetary, since $z(x; \delta) > 0$ for all $x \in \Xi$.

(ii) Immediate from (26), and the fact that $L[\Lambda(s)] = L(\delta)$ for all $s$ in the proposed equilibrium.

(iii) The equilibrium state-by-state gross rate of change in fruit price is $1 + \pi(s', s) = \frac{\phi^m(s)}{\phi^m(s')}$, which becomes (43) after substituting (42). The equilibrium state-by-state gross rate of change in the price of general goods is $1 + \pi_g(s', s) = \frac{\lambda_m(s)}{\lambda_m(s')}$, where $\lambda_m(s) = U'(x) \phi^m(s)$, which gives (44).

(iv) (a) This was shown (i).

(b) From (39) and part (a),

$$z(x; \delta') = \delta' q^* - \lambda(x; \delta') < \delta q^* - \lambda(x; \delta) = z(x; \delta).$$

(c) Immediate from (41) and part (a).

(d) Immediate from (42) and part (b).

(e) Immediate from (ii) and $L'(\delta) = \alpha u''(\delta q^*) q^* < 0$.

(f) With (44), and Jensen’s Inequality,

$$\int [1 + \pi_g(x', x; \delta)] dF(x', x) = \beta L(\delta) \int \frac{z(x'; \delta) dF(x'; x) dF(x', x)}{z(x'; \delta)} \geq \beta L(\delta),$$

with strict inequality unless $z(x; \delta)$ is a degenerate random variable.

(v) $\lim_{\delta \to 1} z(x; \delta) = q^* - \lambda(x; 1)$, where $\lambda(x; 1)$ is the unique continuous, bounded, and strictly positive solution to (15), i.e., $\lambda^s(x)$ in Proposition 2. Thus, $z(x; 1)$ is the same function as $z^s(x)$ in Proposition 2. Finally, from (40),

$$\mu(x; 1) = \beta L(1) \frac{z(x'; 1) dF(x'; x)}{z(x; 1)} = \beta \int z^s(x') dF(x', x) z^s(x),$$

32
is the same as $\mu^*(x)$ described in Proposition 2. ■

**Proof of Corollary 1.** The expressions (45)–(49) are obtained from setting $F(x', x) = F(x)$ in Proposition 3. Notice that the assumption $B \leq [1 - \beta L(\delta_0)] \delta_0 q^*$ implies

$$\sup_{x \in \Xi} |xU'(x)| < \sup_{x \in \Xi} |U(x)| \leq B \leq [1 - \beta L(\delta_0)] \delta_0 q^* \leq [1 - \beta L(\delta)] \delta q^*$$

for all $\delta \in [\delta_0, 1]$, so just as in Proposition 3, $\mu(x; \delta) > 0$, for all $x \in \Xi$ and $\phi^m(s; \delta) > 0$ for all $s = (x, M) \in \Xi \times \mathbb{R}^+$. The value of the equilibrium money holdings in state $s = (x, M)$, under the policy $\mu(x; \delta)$ in the equilibrium described in the statement, is

$$z(x; \delta) = U'(x) \phi^m(s; \delta) M = \delta q^* - \frac{\beta L(\delta)}{1 - \beta L(\delta)} \int x'U'(x') dF(x') - xU'(x) > 0$$

for all $x \in \Xi$. ■

**Proof of Corollary 2.** The expressions (51)–(55) are obtained from (40)–(44) by substituting $U'(x) = 1/x$. For any $\delta \in [\delta_0, 1]$, the assumption $1 < [1 - \beta L(\delta_0)] \delta_0 q^*$ guarantees the equilibrium is indeed monetary, since the assumption implies that $\phi^m(s; \delta) > 0$ for all $s = (x, M) \in \Xi \times \mathbb{R}^+$ (money is valued), and that $z(x; \delta) = \delta q^* - \frac{1}{1 - \beta L(\delta)} > 0$ for all $x \in \Xi$ (the value of the equilibrium money holdings under the policy $\mu(x; \delta)$ in the equilibrium described in the statement is always positive). ■
References


Some Results on the Optimality and Implementation of the Friedman Rule in the Search Theory of Money

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Abstract
I characterize a large family of monetary policies that implement Milton Friedman’s prescription of zero nominal interest rates in a monetary search economy with multiple assets and aggregate uncertainty. This family of optimal policies is defined by two properties: (i) the money supply must be arbitrarily close to zero for an infinite number of dates, and (ii) asymptotically, on average (over the dates when fiat money plays an essential role), the growth rate of the money supply must be at least as large as the rate of time preference.

Keywords: Friedman rule, interest rates, liquidity, monetary policy, search
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Our final rule for the optimum quantity of money is that it will be attained by a rate of price deflation that makes the nominal rate of interest equal to zero.


1 Introduction

Milton Friedman’s prescription that monetary policy should induce a zero nominal interest rate in order to lead to an optimal allocation of resources, has come to be known as the Friedman rule. The cost of producing real balances is zero to the government, so the optimum quantity of real balances should be such that the marginal benefit is zero to the economic agents. Friedman’s insight is so basic, so fundamental, that one would hope for his prescription to be valid regardless of the particular stance one assumes about the role that money plays in the economy. For some time now, the Friedman rule has been known to be optimal in competitive reduced-form monetary models under fairly broad conditions, whether money is introduced as an argument in the agents’ utility functions, or through a cash-in-advance constraint. Recent developments in the Search Theory of Money have found that (absent extraneous, nonmonetary distortions) Friedman’s rule is also optimal in environments where money is valued as a medium of exchange, and the mechanism of exchange is modeled explicitly.

One simple monetary policy that typically implements the Friedman rule is to contract the money supply at the rate of time preference. To date, this particular implementation of the Friedman rule has been the only one explored in search models of money—to the point that one may be led to the conclusion that contracting the money supply at the rate of time preference in fact is the Friedman rule in this class of models.

In this paper I characterize a large family of monetary policies that are necessary and sufficient to implement zero nominal interest rates (in the sense that these policies are consistent with the existence of a monetary equilibrium with zero nominal interest rates) in a version of Lagos and Wright (2005), augmented to allow for aggregate liquidity shocks and a real financial asset that can be used as means of payment the same way money can. This family of optimal policies is defined by two properties: (i) the money supply must be arbitrarily close to zero for an infinite number of dates, and (ii) asymptotically, on average over the dates when fiat money plays an essential role, the growth rate of the money supply must be at least as large as the rate of time preference. For parametrizations such that the random value of the real asset is
too low in every state of the world to satisfy the agents' liquidity needs, the family of optimal policies that I identify specializes to the class of monetary policies that Wilson (1979) and Cole and Kocherlakota (1998) have shown to be necessary and sufficient to implement the Friedman rule in the context of deterministic cash-in-advance economies. Given what was already known about the optimality of the Friedman rule in competitive reduced-form models, recent work in the Search Theory of Money has underscored the robustness of Friedman’s basic insight and the ensuing prescription of zero nominal interest rates. The findings I report here, underscore the robustness of the characterization of a large class of monetary policies that implement Friedman’s prescription.

2 The model

The model builds on Lagos and Wright (2005) and Lucas (1978). Time is discrete, and the horizon infinite. There is a $[0, 1]$ continuum of infinitely lived agents. Each time period is divided into two subperiods where different activities take place. There are three nonstorable and perfectly divisible consumption goods at each date: *fruit*, *general goods*, and *special goods*. (“Nonstorable” means that the goods cannot be carried from one subperiod to the next.) Fruit and general goods are homogeneous goods, while special goods come in many varieties. The only durable commodity in the economy is a set of “Lucas trees.” The number of trees is fixed and equal to the number of agents. Trees yield (the same amount of) a random quantity $d_t$ of fruit in the second subperiod of every period $t$. The realization of the fruit dividend $d_t$ becomes known to all at the beginning of period $t$ (when agents enter the first subperiod). Production of fruit is entirely exogenous: no resources are utilized and it is not possible to affect the output at any time. The motion of $d_t$ is described by a sequence of functions $F_t(s_{t+1}, s^t) = \Pr (d_{t+1} \leq s_{t+1} | d^t = s^t)$, where $d^t$ denotes a history of realizations of fruit dividends through period $t$, i.e., $d^t = (d_t, d_{t-1}, \ldots, d_0)$. For each fixed $s^t$, $F_t(\cdot, s^t)$ is a distribution function with support $\Xi \subseteq (0, \infty)$.

In each subperiod, every agent is endowed with $\bar{n}$ units of time which can be employed as
labor services. In the second subperiod, each agent has access to a linear production technology that transforms labor services into general goods. In the first subperiod, each agent has access to a linear production technology that transforms his own labor input into a particular variety of the special good that he himself does not consume. This specialization is modeled as follows. Given two agents $i$ and $j$ drawn at random, there are three possible events. The probability that $i$ consumes the variety of special good that $j$ produces but not vice-versa (a single coincidence) is denoted $\alpha$. Symmetrically, the probability that $j$ consumes the special good that $i$ produces but not vice-versa is also $\alpha$. In a single-coincidence meeting, the agent who wishes to consume is the buyer, and the agent who produces, the seller. The probability neither wants the good that the other can produce is $1 - 2\alpha$, with $\alpha \leq 1/2$. In contrast to special goods, fruit and general goods are homogeneous, and hence consumed (and in the case of general goods, also produced) by all agents.

In the first subperiod, agents participate in a decentralized market where trade is bilateral (each meeting is a random draw from the set of pairwise meetings), and the terms of trade are determined by bargaining. The specialization of agents over consumption and production of the special good combined with bilateral trade, give rise to a double-coincidence-of-wants problem in the first subperiod. In the second subperiod, agents trade in a centralized market. Agents cannot make binding commitments, and trading histories are private in a way that precludes any borrowing and lending between people, so all trade—both in the centralized and decentralized markets—must be quid pro quo.

Each tree has outstanding one durable and perfectly divisible equity share that represents the bearer’s ownership, and confers him the right to collect the fruit dividends. There is a second financial asset, money, which is intrinsically useless (it is not an argument of any utility or production function), and unlike equity, ownership of money does not constitute a right to collect any resources. Money is issued by a “government” that at $t = 0$ commits to a monetary policy, represented by a sequence of positive real-valued functions, $\{\mu_t\}_{t=0}^\infty$. Given an initial stock of money, $M_0 > 0$, a monetary policy induces a money supply process, $\{M_t\}_{t=0}^\infty$, via $M_{t+1} = \mu_t \left( d^t \right) M_t$.\footnote{In general, a monetary policy induces a stochastic process $\{M_t\}_{t=0}^\infty$, i.e., a collection of random variables, $M_t$, defined on an appropriate probability space (see the appendix for more details). As a special case, a deterministic monetary policy, that is, the case where $\{\mu_t\}_{t=0}^\infty$ is a sequence of positive constants, induces a deterministic money supply process, i.e., a deterministic sequence, $\{M_t\}_{t=0}^\infty$. Proposition 1 and Proposition 2 are proven for a general stochastic money supply process, while the characterization in Proposition 3 focuses on the case where $\{M_t\}_{t=0}^\infty$ is a deterministic money supply process. See Lagos (2008) for a characterization of a large class of stochastic monetary policies.} The government injects or withdraws money via lump-sum transfers or
taxes in the second subperiod of every period, i.e., along every sample path, $M_{t+1} = M_t + T_t$, where $T_t$ is the lump-sum transfer (or tax, if negative). All assets are perfectly recognizable, cannot be forged, and can be traded among agents both in the centralized and decentralized markets.\textsuperscript{3} At $t = 0$ each agent is endowed with $a_s^0$ equity shares and $a_m^0$ units of fiat money.

Let the utility function for special goods, $u : \mathbb{R}^+ \to \mathbb{R}^+$, and the utility function for fruit, $U : \mathbb{R}^+ \to \mathbb{R}^+$, be continuously differentiable, increasing, and strictly concave, with $u(0) = U(0) = 0$, and let $U$ be bounded. Let $-n$ be the utility from working $n$ hours in the first subperiod. Also, suppose there exists $q^* \in (0, \infty)$ defined by $u'(q^*) = 1$, with $q^* \leq \bar{n}$. Let both, the utility for general goods, and the disutility from working in the second subperiod, be linear. The agents rank consumption and labor sequences according to

$$\liminf_{T \to \infty} E_0 \left\{ \sum_{t=0}^{T} \beta^t [u(q_t) - nt + U(c_t) + y_t - h_t] \right\},$$

where $\beta \in (0, 1)$, $q_t$ and $n_t$ are the quantities of special goods consumed and produced in the decentralized market, $c_t$ denotes consumption of fruit, $y_t$ consumption of general goods, $h_t$ the hours worked in the second subperiod, and $E_t$ is an expectations operator conditional on the information available to the agent at time $t$, defined with respect to the matching probabilities and the probability measure over sequences of dividends and money supplies induced by the sequence of transition functions, $\{F_t\}_{t=0}^{\infty}$, and the monetary policy, $\{\bar{\mu}_t\}_{t=0}^{\infty}$.\textsuperscript{4}

3 Equilibrium

Let $a_t = (a_s^t, a_m^t)$ denote the portfolio of an agent who holds $a_s^t$ shares and $a_m^t$ dollars. Let $W_t(a_t)$ and $V_t(a_t)$ be the maximum attainable expected discounted utility of an agent when he enters the centralized, and decentralized market, respectively, at time $t$ with portfolio $a_t$. money supply processes that implement the Friedman rule in a stationary version of this environment.

\textsuperscript{3}Lagos and Rocheteau (2008) was the first paper to extend Lagos and Wright (2005) to allow for another asset that competes with money as a medium of exchange. Lagos (2006) considers a real version of Lagos and Wright (2005) with aggregate uncertainty, in which equity shares and government bonds can serve as means of payment, and quantifies the extent to which a liquidity premium can help to explain the equity premium and the risk-free rate puzzles. While the formulation I am studying in this paper has only two financial assets, equity and money, it would not be difficult to extend the results to an environment with a richer asset structure.

\textsuperscript{4}I follow Brock (1970) and use this “overtaking criterion” to rank sequences of consumption and labor because $\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t [u(q_t) - nt + U(c_t) + y_t - h_t]$ may not be well defined for some feasible sequences. The same criterion was adopted by Wilson (1979) and Cole and Kocherlakota (1998) in their studies of competitive monetary economies subject to cash-in-advance constraints, and by Green and Zhou (2002) in their study of dynamic monetary equilibria in a random matching economy.
Then,

\[ W_t (a_t) = \max_{c_t, y_t, h_t, \lambda_t, \phi_t, a_{t+1}} \{ U (c_t) + y_t - h_t + \beta E_t V_{t+1} (a_{t+1}) \} \]

subject to:

\[ c_t + w_t y_t + \phi_t^s a_t^s + \phi_t^m a_t^m = (\phi_t^s + d_t) a_t^s + \phi_t^m (a_t^m + T_t) + w_t h_t \]

\[ 0 \leq c_t, \ 0 \leq h_t \leq \bar{n}, \ 0 \leq a_{t+1}. \]

The agent chooses consumption of fruit \((c_t)\), consumption of general goods \((y_t)\), labor supply \((h_t)\), and an end-of-period portfolio \((a_{t+1})\). Fruit is used as numéraire: \(w_t\) is the relative price of the general good, \(\phi_t^s\) is the (ex-dividend) price of a share, and \(1/\phi_t^m\) the dollar price of fruit. Substitute the budget constraint into the objective and rearrange to arrive at:

\[ W_t (a_t) = \lambda_t a_t + \tau_t + \max_{c_t \geq 0} [U (c_t) - c_t] + \max_{a_{t+1} \geq 0} [\phi_t^s a_t^s + \phi_t^m (a_t^m + T_t) + w_t h_t], \]

where \(\tau_t = \lambda_t^m T_t\), \(\phi_t = (\phi_t^s, \phi_t^m)\), and \(\lambda_t = (\lambda_t^s, \lambda_t^m)\), with

\[ \lambda_t^s \equiv \frac{1}{w_t} (\phi_t^s + d_t) \quad \text{and} \quad \lambda_t^m \equiv \frac{1}{w_t} \phi_t^m. \]

Let \([q_t (a, \tilde{a}), p_t (a, \tilde{a})]\) denote the terms at which a buyer who owns portfolio \(a\) trades with a seller who owns portfolio \(\tilde{a}\), where \(q_t (a, \tilde{a}) \in \mathbb{R}^+\) is the quantity of special good traded, and \(p_t (a, \tilde{a}) = [p_t^s (a, \tilde{a}), p_t^m (a, \tilde{a})] \in \mathbb{R}^+ \times \mathbb{R}^+\) is the transfer of assets from the buyer to the seller (the first argument is the transfer of equity). Consider a meeting in the decentralized market of period \(t\), between a buyer with portfolio \(a_t\) and a seller with portfolio \(\tilde{a}_t\). The terms of trade, \((q_t, p_t)\), are determined by Nash bargaining where the buyer has all the bargaining power:

\[ \max_{q_t, p_t \leq a_t} [u (q_t) + W_t (a_t - p_t) - W_t (a_t)] \quad \text{s.t.} \quad W_t (\tilde{a}_t + p_t) - q_t \geq W_t (\tilde{a}_t). \]

The constraint \(p_t \leq a_t\) indicates that the buyer in a bilateral meeting cannot spend more assets than he owns. Since \(W_t (a_t + p_t) - W_t (a_t) = \lambda_t p_t\), the bargaining problem is

\[ \max_{q_t, p_t \leq a_t} [u (q_t) - \lambda_t p_t] \quad \text{s.t.} \quad \lambda_t p_t - q_t \geq 0. \]

If \(\lambda_t a_t \geq q^*\), the buyer buys \(q_t = q^*\) in exchange for a vector \(p_t\) of assets with real value \(\lambda_t p_t = q^* \leq \lambda_t a_t\). Else, he pays the seller \(p_t = a_t\), in exchange for \(q_t = \lambda_t a_t\). Hence, the quantity of output exchanged is \(q_t (a_t, \tilde{a}_t) = \min (\lambda_t a_t, q^*) \equiv q (\lambda_t a_t)\), and the real value of the portfolio used as payment is \(\lambda_t p_t (a_t, \tilde{a}_t) = q (\lambda_t a_t)\).
With the bargaining solution and the fact that \( W_t (a_t) \) is affine, the value of search to an agent who enters the decentralized market of period \( t \) with portfolio \( a_t \) can be written as

\[
V_t (a_t) = S (\lambda_t a_t) + W_t (a_t),
\]

where \( S (x) \equiv \alpha\{u[q(x)] - q(x)\} \) is the expected gain from trading in the decentralized market.\(^5\) Substitute (6) into (4) to arrive at

\[
\begin{align*}
W_t (a_t) &= \lambda_t a_t + \tau_t + \max_{c_t \geq 0} \left[ U (c_t) - \frac{c_t}{w_t} \right] \\
&\quad + \max_{a_{t+1} \geq 0} \left\{ -\frac{\phi_t a_{t+1}}{w_t} + \beta E_t [S (\lambda_{t+1} a_{t+1}) + W_{t+1} (a_{t+1})] \right\}.
\end{align*}
\]

The agent’s problem consists of choosing a feasible plan \( \{c_t, x_t, a_{t+1}^s, a_{t+1}^m\}_{t=0}^\infty \) that maximizes (1), taking as given the money supply process \( \{M_t\}_{t=0}^\infty \), the bargaining protocol, and the sequence of price functions \( \{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty \). For each \( t \), each element of the plan, \( c_t \) (fruit consumption), \( x_t \) (production of general goods minus consumption of general goods), \( a_{t+1} = (a_{t+1}^s, a_{t+1}^m) \) (equity and money holdings), is a function of the history of dividends and money supplies, and similarly for the price functions.\(^6\) The plan is feasible if it satisfies the initial conditions, and (2) and (3) in every history.\(^7\) The functional equation (7) is a convenient representation of the agent’s problem. The following result shows that the sequences of solutions for fruit consumption and asset holdings induced by the maximization problems in (7) for \( t = 0, 1, ... \) that satisfy certain boundedness conditions, solve the agent’s time-0 optimization problem.

**Proposition 1** Given a money supply process \( \{M_t\}_{t=0}^\infty \) and a sequence of price functions \( \{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty \), a feasible plan \( \{c_t, a_{t+1}^s, a_{t+1}^m\}_{t=0}^\infty \) is optimal for the agent from \( t = 0 \), given

\( ^5\) Note that \( S \) is twice differentiable almost everywhere, with \( S' (x) \geq 0 \) and \( S'' (x) \leq 0 \) (both inequalities are strict for \( x < q^* \)), and that \( \frac{\partial u (\lambda a_t)}{\partial a_t} = \lambda_t^* \) if \( \lambda_t a_t < q^* \), \( \frac{\partial u (\lambda a_t)}{\partial a_t} = 0 \) if \( \lambda_t a_t \geq q^* \), and \( \frac{\partial u (\lambda a_t)}{\partial a_t} = \lambda_t^* \) is the expected gain from trading in the decentralized market.

\( ^6\) See the appendix for a formal description of the time-0 infinite-horizon problem.

\( ^7\) For each history, given \( c_t, a_{t+1}^s, \) and \( a_{t+1}^m \), the money supply process \( \{M_t\}_{t=0}^\infty \), and the price functions \( \{w_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty \), the net production of general goods, \( x_t \), is immediate from (2), so it will be left implicit in the definition of equilibrium and the analysis hereafter.
initial conditions \( \mathbf{a}_0 = (a_{0}, a_{0}^{m}) \) and \( d_0 \), if and only if

\[
U'(c_t) - \frac{1}{w_t} \leq 0, \quad " = " \text{ if } c_t > 0
\]

(8) implies

\[
-\frac{1}{w_t} \phi^s_t + \beta E_t \left\{ \left[ 1 + S'(\lambda^s_{t+1} a_{t+1}^{s}) \right] \lambda^s_{t+1} \right\} = 0, \quad " = " \text{ if } a_{t+1}^s > 0
\]

(9) implies

\[
-\lambda^m_t + \beta E_t \left\{ \left[ 1 + S'(\lambda^m_{t+1} a_{t+1}^{m}) \right] \lambda^m_{t+1} \right\} = 0, \quad " = " \text{ if } a_{t+1}^m > 0
\]

(10) implies

\[
\lim_{t \to \infty} E_0 \left[ \beta^t \frac{1}{w_t} \phi^s_{t+1} a_{t+1}^{s} \right] = 0
\]

(11) implies

\[
\lim_{t \to \infty} E_0 \left[ \beta^t \frac{1}{w_t} \phi^m_{t+1} a_{t+1}^{m} \right] = 0.
\]

(12) implies

\[ w_t = 1/U'(d_t), \]

and once \( \{ \phi^s_t, \phi^m_t \} \) is found, \( \{ q_t \} = \{ \lambda \} \) is determined, i.e., \( q_t = \min (\lambda_t a_t, q^*) \) and \( \lambda_t p_t = q_t \); and (iii) the centralized market clears, i.e., \( c_t = d_t \), and \( a_{t+1}^s = 1 \). The equilibrium is monetary if \( \phi^m_{t+1} > 0 \) for all \( t \), and in this case the money-market clearing condition is \( a_{t+1}^m = M_{t+1} \).

The market-clearing conditions immediately give the equilibrium allocations

\[
\{ c_t, a_{t+1}^s, a_{t+1}^m \}_{t=0}^{\infty} = \{ d_t, 1, M_{t+1} \}_{t=0}^{\infty},
\]

(8) implies \( w_t = 1/U'(d_t) \), and once \( \{ \phi^s_t, \phi^m_t \} \) has been found, \( \{ q_t \} = \{ \lambda \} \) is determined, i.e., \( q_t = \min (\lambda_{t+1}, q^*) \) where \( \Lambda_{t+1} = \Lambda^s_{t+1} + \Lambda^m_{t+1} M_{t+1} \). Therefore, given a money supply process \( \{ M_t \} \), and letting \( L(\Lambda_{t+1}) = [1 + S'(\Lambda_{t+1})] \), a monetary equilibrium can be summarized by a sequence \( \{ \phi^s_t, \phi^m_t \} \) that satisfies

\[
U' (d_t) \phi^s_t = \beta E_t \left[ L(\Lambda_{t+1}) U'(d_{t+1}) (\phi^s_{t+1} + d_{t+1}) \right]
\]

(13)

\[
\lambda^m_t = \beta E_t \left[ L(\Lambda_{t+1}) \lambda^m_{t+1} \right]
\]

(14)

\[
\lim_{t \to \infty} E_0 \left[ \beta^t U'(d_t) \phi^s_t \right] = 0
\]

(15)

\[
\lim_{t \to \infty} E_0 \left[ \beta^t \lambda^m_t M_{t+1} \right] = 0.
\]

(16)

There are two assets in this model: equity shares and fiat money. However, to state the results that follow, it will be convenient to be able to refer to a notion of nominal interest rate. In order to derive an expression for the “shadow” nominal interest rate, imagine there existed
an additional asset in this economy, an illiquid nominal bond, i.e., a one-period risk-free bond that pays a unit of money in the centralized market, and which cannot be used in decentralized exchange. Let $\phi^n_t$ denote the price of this nominal bond. In equilibrium, this price must satisfy

$$U'(d_t) \phi^n_t = \beta E_t [U'(d_{t+1}) \phi^n_{t+1}].$$

Since $\phi^n_t / \phi^m_t$ is the dollar price of a nominal bond, $i_t = \frac{\phi^n_t}{\phi^m_t} - 1$ is the nominal interest rate. In a monetary equilibrium,

$$i_t = \frac{E_t [L (\Lambda_{t+1}) \lambda^n_{t+1}]}{E_t (\lambda^m_{t+1})} - 1. \quad (17)$$

### 4 Optimal monetary policy

In this section I consider the problem of choosing an optimal monetary policy. The Pareto optimal allocation can be found by solving the problem of a social planner who wishes to maximize average (equally-weighted across agents) expected utility. The planner chooses a contingent plan $\{c_t, q_t, n_t, y_t, h_t\}_{t=0}^\infty$ subject to the resource constraints, i.e., $y_t \leq h_t$, and $q_t \leq n_t$ for those agents who are matched in the first subperiod of period $t$, and $q_t = n_t = 0$ for those agents who are not. After imposing these constraints (with equality, with no loss of generality), the planner’s problem becomes

$$\max_{\{c_t, q_t\}_{t=0}^\infty} \lim_{T \to \infty} E_0 \left\{ \sum_{t=0}^T \beta^t \left[ \alpha \left[ w(q_t) - q_t \right] + U(c_t) \right] \right\}$$

subject to $0 \leq q_t$, and $0 \leq c_t \leq d_t$, and the initial condition $d_0 \in \Xi$. Here, $E_0$ denotes the expectation with respect to the probability measure over sequences of dividend realizations induced by $\{F_t\}_{t=0}^\infty$. The solution is to set $\{c_t, q_t\}_{t=0}^\infty = \{d_t, q^*\}_{t=0}^\infty$. From Definition 1, it is clear that the equilibrium consumption of fruit always coincides with the efficient allocation. However, the equilibrium allocation has $q_t \leq q^*$, which may hold with strict inequality in some states. That is, in general, consumption and production in the decentralized market may be too low in a monetary equilibrium.\(^8\)

**Proposition 2** Equilibrium quantities in a monetary equilibrium are Pareto optimal if and only if $i_t = 0$ almost surely (a.s.) for all $t$.

The following proposition, establishes two results. The first, is that a deterministic money supply process can suffice to implement a zero nominal rate in every state of the world, even

---

\(^8\)This is a standard result in the literature, see Lagos and Wright (2005).
though liquidity needs are stochastic in this environment (because equity, whose value is stochastic, can be used alongside money as means of payment). The second, is that even within the class of deterministic monetary policies, there is a large family of policies that implement the Pareto optimal equilibrium (i.e., there exists a monetary equilibrium with zero nominal rates in every state of the world under the policy). Versions of the second result have been proven by Wilson (1979) and Cole and Kocherlakota (1998), for deterministic competitive economies with cash-in-advance constraints that are imposed on agents every period with probability one. Before stating the proposition, it is convenient to introduce some notation. Let

\[ \lambda_t^{**} = U'(d_t)(\phi_t^{**} + d_t), \]

where

\[ \phi_t^{**} = E_t \sum_{j=1}^{\infty} \beta^j U'(d_t+j)d_{t+j}, \] (18)

and let \( T \) denote the set of dates, \( t \), for which \( q^* - \lambda_t^{**} > 0 \) holds with probability \( \pi_t > 0 \).

**Proposition 3** Assume that \( \inf_{t \in T} \pi_t > 0 \). A monetary equilibrium with \( \nu_t = 0 \) a.s. for all \( t \) exists under a deterministic money supply process \( \{M_t\}_{t=0}^{\infty} \) if and only if the following two conditions hold:

\[ \liminf_{t \to \infty} M_t = 0 \] (19)
\[ \inf_{t \in T} M_t \beta^{-t} > 0 \text{ if } T \neq \emptyset. \] (20)

Conditions (19) and (20) are rather unrestrictive asymptotic conditions. The first one requires that the money supply be arbitrarily close to zero for an infinite number of dates, or equivalently, that there exists some subsequence of dates \( \{t_1, t_2, \ldots\} \), such that \( \lim_{n \to \infty} M_{t_n} = 0 \). The second condition requires that asymptotically, on average over the set of dates \( T \) when fiat money plays an essential role, the growth rate of the money supply must be at least as large as the rate of time preference.\(^9\)

The simple class of policies of contracting the money supply at a constant rate, e.g., \( M_t = \gamma^t M_0 \) for \( \gamma \in [\beta, 1) \), satisfies (19) and (20), and hence is consistent with a monetary equilibrium with zero nominal rates in every state. But many other policies are as well. For example, for

\(^9\)To get some intuition on (20), consider an economy with \( \pi_t = 1 \) for all \( t \) under an arbitrary deterministic monetary policy, i.e., suppose that \( \{\mu_t\}_{t=0}^{\infty} \) is a positive sequence of real numbers such that \( M_t = (\bar{\mu}_t)^t M_0 \), where \( \bar{\mu}_t = \left( \prod_{i=0}^{t-1} \mu_i \right)^{1/t} \) is the geometric average of the growth rate of the money supply through time \( t \). In this case, condition (20) is equivalent to \( \liminf_{t \to \infty} (\bar{\mu}_t/\beta)^t > 0 \).
$b > 0$ sufficiently small, consider

$$M_{t+1} = \gamma^t \left[ 1 + b \sin(t) \right] M_0$$

(21)

for any $\gamma \in (\beta, 1)$. Under (21) the money supply follows deterministic cycles of expansion and contraction forever, and may even contract at a rate larger than $\beta$ infinitely often, yet this policy implements a monetary equilibrium where the nominal rate is zero in all states.

Consider a deterministic policy $\{M_t\}_{t=0}^{\infty}$ that satisfies the conditions in Proposition 3. Then, there is a monetary equilibrium with $\phi^*_t = \phi^*_t$ and $\lambda^*_t = \beta^{-t} \lambda^*_0$, where $\lambda^*_0 > 0$ is a constant that can be chosen arbitrarily, subject to the additional restriction that

$$\lambda^*_0 \geq q^* - \lambda^*_0 \phi^*_t$$

for all $t \in T$.

In other words, the value of money, $\lambda^*_m$, and hence the price level (e.g., the nominal price of fruit, $1/\phi^*_m$, and the nominal price of general goods, $1/\left[U'(d_t) \phi^*_m]\right]$) is indeterminate under an optimal monetary policy. In this monetary equilibrium, the inflation rate (e.g., in the price of general goods) is

$$\frac{\lambda^*_t}{\lambda^*_t} = \beta,$$

which is independent of the path of the money supply. This means that the monetary equilibrium with zero nominal interest rates just described, could be obtained, for instance, both under $M_{t+1} = \beta M_0$ or under (21), but the inflation rate is the same regardless of which monetary policy is actually followed. Proposition 3 implies that the quantity theory is in general not valid under an optimal monetary policy. (In the equilibrium just described, the quantity theory would no be falsified, however, if the monetary policy was $M_t = \beta M_0$.) This feature of an optimal monetary policy was emphasized by Cole and Kocherlakota (1998) in the context of their deterministic cash-in-advance economy.

Assume away the Lucas trees, and this economy reduces to Lagos and Wright (2005) with buyer-takes-all bargaining. In that paper and in all the subsequent literature, the monetary policy analysis focuses exclusively on equilibria where real money balances are constant, and the money supply grows or declines at a constant rate.\(^\text{10}\) The usual finding is that the policy of contracting the money supply at a constant rate equal to the rate of time preference is optimal.\(^\text{11}\)

\(^\text{10}\) Lagos and Wright (2003) analyze dynamic equilibria with real balances that vary over time, but do not study monetary policy (the money supply is kept constant to focus on dynamics due exclusively to beliefs).

\(^\text{11}\) This policy implements $q_t = q^*$ in every meeting with buyer-takes-all bargaining. Otherwise, the policy is still optimal among the feasible class of policies considered, but implements $q_t < q^*$. 
This policy satisfies conditions (19) and (20), and in an equilibrium with constant real balances, it implies that $M_{t+1}/M_t = \phi^m_t/\phi^m_{t+1} = \beta$, and that real balances equal $\phi^m_t M_t = k q^*$ for all $t$, so $\phi^m_t = k q^*/M_t$ for all $t$, where $k \geq 1$ is an arbitrary constant. But as a simple corollary of Proposition 3, this is not the only optimal monetary policy in that model. For example, the policy $M_{t+1}/M_t = \gamma \in (\beta, 1)$ (together with the given $M_0$), is also an optimal policy, since it is consistent with a monetary equilibrium with zero nominal rates, e.g., $\phi^m_t = \beta^{-t} k q^*/M_0$, and real balances $\phi^m_t M_t = (\gamma/\beta)^t k q^*$ that are growing over time ($k \geq 1$ is again an arbitrary constant). The family of monetary policies that implement equilibria where nominal interest rates are zero is large. However, if one chooses to constrain the set of policies to those where the money supply grows at a constant rate, and to restrict attention to equilibria with constant real money balances, then $M_{t+1}/M_t = \beta$ is the only policy in the family defined by conditions (19) and (20).

5 Conclusion

I have formulated a fairly general version of a prototypical search-based monetary model in which money coexists with equity—a financial asset that yields a risky real return. In this formulation, money is not assumed to be the only asset that must, nor the only asset that can, play the role of a medium of exchange: nothing in the environment prevents agents from using equity along with money, or instead of money, as means of payment. Since the equity share is a claim to a risky aggregate endowment, the fact that agents can use equity to finance purchases implies that they face aggregate liquidity risk, in the sense that in some states of the world, the value of equity holdings may turn out to be too low relative to what would be needed to carry out the transactions that require a medium of exchange. This is a natural context to study the role of money and monetary policy in providing liquidity to lubricate the mechanism of exchange. The model could be augmented to include other types of aggregate uncertainty. For example, one could incorporate aggregate productivity shocks to the technology used to produce general goods, but the main results would not be affected. (The formulation that I have studied is isomorphic to one with aggregate productivity shocks to the technology used to produce special goods.)

An implication of Proposition 3 is that even in a simple deterministic economy, it would be impossible for someone with access to a finite time-series for the path of the money supply, to determine whether an optimal monetary policy is being followed. On the other hand, a
single observation of a positive nominal rate would be definitive evidence of a deviation from an optimal monetary policy. According to Proposition 3, there is a large family of monetary policies that are necessary and sufficient to *weakly implement* zero nominal interest rates, in the sense that every policy in the family is consistent with the existence of a monetary equilibrium with zero nominal interest rates. This result leaves open for future research the question of *unique implementation of monetary equilibrium*: Are there monetary policies in the optimal class under which the equilibrium with zero nominal rates at all dates and in all states is the unique monetary equilibrium? It is not difficult to find policies that implement zero nominal rates weakly but not uniquely. Notice that even if one were able to find a family of policies that imply that a monetary equilibrium with zero nominal rates exists, and that it is the unique monetary equilibrium, the question of *unique implementation of equilibrium* would still require one to deal with the fact that a nonmonetary equilibrium will always exist.

Throughout the paper I have emphasized the similarities between my results and those that have been established for competitive economies subject to cash-in-advance constraints, so at this point I should perhaps comment on some of the differences. The analogous results for cash-in-advance economies, e.g., those obtained by Wilson (1979) and by Cole and Kocherlakota (1998), have been established in environments with no aggregate uncertainty, and where it is assumed that money must be used in order to purchase a certain good (with probability one in every period). In contrast, I have proved Propositions 2 and 3 for search-based environments with aggregate uncertainty, and where money may not be used or needed as a medium of exchange in some periods, for some realizations of the aggregate uncertainty.

An implication of Proposition 3 is that even if agent’s liquidity needs are stochastic, one need not look beyond the class of deterministic monetary policies to implement the optimum. In particular, it is not necessary to resort to (state-contingent) monetary policy rules if the purpose is to implement zero nominal rates. To some, the failure of the quantity theory and the ensuing price-level indeterminacy may seem unappealing if the model is to be used for applied research. One way to address these concerns, would be to characterize the family of policies that implement a constant but positive nominal interest rate, and then study the behavior of

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12 For example, in the pure monetary economy discussed in the end of Section 4, I pointed out that the monetary policy $M_{t+1}/M_t = \gamma \in (\beta, 1)$ for all $t$, is consistent with a monetary equilibrium with zero nominal rates and real balances that grow over time. However, under this monetary policy there also exists a monetary equilibrium with constant real balances $\bar{z} \in (0, q^*)$ and a constant nominal interest rate $i = \alpha [u'(\bar{z}) - 1] > 0$. (This last equilibrium exists provided $u'(0) > 1 + \frac{2c - d}{\alpha z^*}$, and in this case, $\bar{z}$ solves $u'(\bar{z}) = 1 + \frac{2c - d}{\alpha z^*}$.)
the limiting economy as the target nominal rate approaches zero. The optimal prescription for monetary policy—the Friedman rule—requires: (a) that the nominal interest be constant, and (b) that this constant be zero, so this class of non-optimal policies would represent a perturbation of the Friedman rule along the second dimension. A policy that targets a constant nonzero nominal rate in this stochastic environment, however, will typically have to implement stochastic real balances, which could require a stochastic monetary policy rule. I study these issues in Lagos (2008).
A Appendix

I begin with a formal description of the time-zero infinite-horizon problem faced by an agent in the monetary economy. Let $\omega = \{d_t, M_t\}_{t=0}^{\infty}$ denote a realization of dividends and money supplies, and let $\Omega$ be the set of all such realizations. Let $\omega^t = \{d_k, M_k\}_{k=0}^{t}$ denote a history of dividends and money supplies up to time $t$, and let $\Omega^t$ be the collection of all such histories. Consider the probability space $(\Omega, \mathcal{H}, \mathbb{P})$, where $\mathcal{H}$ is an appropriate $\sigma$-field of subsets of $\Omega$ (e.g., the $\sigma$-field generated by $\Omega^t$ for all finite $t$), and $\mathbb{P}$ is the probability measure on $\mathcal{H}$ induced by the transition functions $\{F_t\}_{t=0}^{\infty}$ and the monetary policy $\{\mu_t\}_{t=0}^{\infty}$. Let $\mathcal{H}^t \subseteq \mathcal{H}$ be a partition of $\Omega$ such that $H^t_\omega \in \mathcal{H}^t$ is a set of histories that coincide until time $t$, i.e., $H^t_\omega = \{\omega \in \Omega : \omega^t = \bar{\omega} \text{ for some } \bar{\omega} \in \Omega^t\}$. The $\sigma$-field generated by $\mathcal{H}^t$, denoted $\mathcal{F}_t$, captures the information available to the investor at time $t$, and the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ represents how this information is revealed over time.

At $t = 0$, the agent takes as given the sequence of $\mathcal{F}_t$–measurable price functions,

$$\{w_t, \phi^s_t, \phi^m_t\}_{t=0}^{\infty},$$

and the sequence of $\mathcal{F}_t$–measurable monetary policy functions, $\{\mu_t\}_{t=0}^{\infty}$, where $w_t : \Omega \to \mathbb{R}^+, \phi^s_t : \Omega \to \mathbb{R}^+$, and $\mu_{t+1} : \Omega \to \mathbb{R}^+$. From (5), notice that $\lambda^s_t : \Omega \to \mathbb{R}^+$ and $\lambda^m_t : \Omega \to \mathbb{R}^+$ are $\mathcal{F}_t$–measurable functions as well. A feasible plan, $\chi = \{c_t, x_t, a^s_t, a^m_t\}_{t=0}^{\infty}$, is a value $(a^s_0, a^m_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ and a sequence of $\mathcal{F}_t$–measurable functions

$$\{c_t, x_t, a^s_{t+1}, a^m_{t+1}\}_{t=0}^{\infty},$$

where $c_t : \Omega \to \mathbb{R}^+$, $a^s_{t+1} : \Omega \to \mathbb{R}^+$, $a^m_{t+1} : \Omega \to \mathbb{R}^+$, and

$$x_t = \frac{1}{w_t} \left[ (\phi^s_t + d_t) a^s_t + \phi^m_t (a^m_t + T_t) - \phi^s_t a^s_{t+1} - \phi^m_t a^m_{t+1} - c_t \right],$$

with $T_t = M_{t+1} - M_t$. Let $\mathcal{A}$ denote the set of all feasible plans.

Let $U^T(\cdot, a_0, \omega_0)$ be the utility functional for the agent from $t = 0$ until $t = T$, given that $a_0 = (a^s_0, a^m_0)$ is the agent’s initial portfolio, and $\omega_0 = (d_0, M_0)$ is the initial condition for the dividend and the money supply. The agent’s utility from following a feasible policy $\chi$ over this period, taking as given the sequence of price functions $\{w_t, \phi^s_t, \phi^m_t\}_{t=0}^{\infty}$, is

$$U^T(\chi, a_0, \omega_0) = E_0 \left\{ \sum_{t=0}^{T} \beta^t \left[ S(\lambda_t a_t) + \lambda_t a_t - \frac{1}{w_t} \phi^s_t a^s_{t+1} \right] \right\} + E_0 \left\{ \sum_{t=0}^{T} \beta^t \left[ U(c_t) - \frac{1}{w_t} c_t \right] \right\} + K_T,$$
where \( K_T \equiv E_0 \left\{ \sum_{t=0}^{T} \beta^t \frac{\phi^m_t}{w_t} T_t \right\} \). The notation \( E_t \) is shorthand for the conditional expectation \( E [\cdot | \mathcal{F}_t] \). With the Law of Iterated Expectations, \( U^T (\chi, a_0, \omega_0) \) can be rearranged as follows:

\[
U^T (\chi, a_0, \omega_0) = S (\lambda_0 a_0) + \lambda_0 a_0 + K_T + E_0 \left\{ \sum_{t=0}^{T-1} \beta^t \left[ U (c_t) - \frac{1}{w_t} c_t \right] \right\} \\
+ E_0 \left\{ \sum_{t=0}^{T-1} \beta^t \left[ \beta E_t S (\lambda_{t+1} a_{t+1}) - \left( \frac{1}{w_t} \phi^s_t - \beta E_t \lambda^s_{t+1} \right) a^s_{t+1} - \left( \lambda^m_t - \beta E_t \lambda^m_{t+1} \right) a^m_{t+1} \right] \right\} \\
- E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\}. \tag{22}
\]

The utility \( U^T (\cdot, a_0, \omega_0) \) associated with the agent’s problem has been defined for an arbitrary sequence of \( \mathcal{F}_t \)-measurable price functions \( \{ w_t, \phi^s_t, \phi^m_t \}_{t=0}^{\infty} \). Some price functions will be inconsistent with an equilibrium, so there is no loss in restricting the analysis of the agent’s problem to a family of functions that excludes such functions. In particular, there is no loss in restricting the analysis to admissible price functions \( \{ w_t, \phi^s_t, \phi^m_t \}_{t=0}^{\infty} \), namely price functions that satisfy the no-arbitrage conditions, \( \beta E_t \lambda^s_{t+1} - \frac{1}{w_t} \phi^s_t \leq 0 \), and \( \beta E_t \lambda^m_{t+1} - \lambda^m_t \leq 0 \) for all \( t \).

Next, define the infinite-horizon utility for the agent from following a feasible plan \( \chi \), by

\[
U (\chi, a_0, \omega_0) = \liminf_{T \to \infty} U^T (\chi, a_0, \omega_0).
\]

Lemma 1 Given admissible price functions \( \{ w_t, \phi^s_t, \phi^m_t \}_{t=0}^{\infty} \),

\[
U (\chi, a_0, s_0) = S (\lambda_0 a_0) + \lambda_0 a_0 + \liminf_{T \to \infty} K_T + E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ U (c_t) - \frac{1}{w_t} c_t \right] \right\} \\
+ E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ \beta E_t S (\lambda_{t+1} a_{t+1}) - \left( \frac{1}{w_t} \phi^s_t - \beta E_t \lambda^s_{t+1} \right) a^s_{t+1} - \left( \lambda^m_t - \beta E_t \lambda^m_{t+1} \right) a^m_{t+1} \right] \right\} \\
- \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\}. \tag{23}
\]

Proof. Let \( S_T = \sum_{t=0}^{T} \beta^t U (c_t) + \sum_{t=0}^{T-1} \beta^{t+1} E_t S (\lambda_{t+1} a_{t+1}) \). \( U \) and \( S \) are nondecreasing, with \( U (0) = S (0) = 0 \), so \( U (c_t) \geq 0 \), and \( S (\lambda_{t+1} a_{t+1}) \geq 0 \) for all \( t \), and therefore \( \{ S_T \}_{T=0}^{\infty} \) is a nondecreasing sequence of nonnegative (extended) real-valued measurable functions, and it has a limit, i.e., \( \lim_{T \to \infty} S_T = \sum_{t=0}^{\infty} \beta^t [ U (c_t) + \beta E_t S (\lambda_{t+1} a_{t+1}) ] \). Then by the Monotone Convergence Theorem (e.g., Theorem 7.8 in Stokey and Lucas, 1989),

\[
\liminf_{T \to \infty} E_0 S_T = \lim_{T \to \infty} E_0 S_T = E_0 \lim_{T \to \infty} S_T. \tag{24}
\]
Let \( S'_T = \sum_{t=0}^{T} \beta^t \frac{1}{w_t} c_t + \sum_{t=0}^{T-1} \beta^t \left[ \frac{1}{w_t} \phi_t^{s} - \beta E_t \lambda_{t+1}^{s} \right] a_{t+1}^s + \left( \lambda_{t}^{m} - \beta E_t \lambda_{t+1}^{m} \right) a_{t+1}^m \). Since each term in this partial sum is nonnegative, \( \{ S'_T \} \) is a nondecreasing sequence, so \( \lim_{T \to \infty} S'_T = \sum_{t=0}^{\infty} \beta^t \left[ \frac{1}{w_t} c_t + \left( \frac{1}{w_t} \phi_t^{s} - \beta E_t \lambda_{t+1}^{s} \right) a_{t+1}^s + \left( \lambda_{t}^{m} - \beta E_t \lambda_{t+1}^{m} \right) a_{t+1}^m \] exists, although it may be \( +\infty \). Then by the Monotone Convergence Theorem,

\[
\liminf_{T \to \infty} E_0 S'_T = \lim_{T \to \infty} E_0 S'_T = E_0 \lim_{T \to \infty} S'_T. \tag{25}
\]

With (24) and (25), take \( \lim\inf \) on both sides of (22) to arrive at (23). \( \blacksquare \)

At \( t = 0 \), the agent takes as given the initial conditions \( a_0 \) and \( \omega_0 \), and a sequence of price functions \( \{ w_t, \phi_t^{s}, \phi_t^{m} \}_{t=0}^{\infty} \), and solves

\[
\max_{\chi \in A} \mathcal{U} (\chi, a_0, \omega_0). 
\]

Proposition 1 characterizes the optimal plan (the maximal plan, in the terminology of Brock, 1970).

**Proof of Proposition 1.** Step (i): First show that (8)–(12) are sufficient for an optimum. Let \( \chi = \{ c_t, x_t, a_{t+1}^s, a_{t+1}^m \}_{t=0}^{\infty} \) be the plan that satisfies (8)–(12), and \( \tilde{\chi} = \{ \tilde{c}_t, \tilde{x}_t, \tilde{a}_{t+1}^s, \tilde{a}_{t+1}^m \}_{t=0}^{\infty} \) be any other feasible plan. Let \( \Delta \equiv \mathcal{U} (\chi, a_0, \omega_0) - \mathcal{U} (\tilde{\chi}, a_0, \omega_0) \), then

\[
\Delta = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ U (c_t) - U (\tilde{c}_t) - \frac{1}{w_t} (c_t - \tilde{c}_t) \right\} \right\} + E_0 \left\{ \sum_{t=0}^{\infty} \beta^{t+1} E_t \left[ S (\lambda_{t+1} a_{t+1}) - S (\lambda_{t+1} \tilde{a}_{t+1}) \right] \right\}
\]

\[
- E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ \frac{1}{w_t} \phi_t^{s} - \beta E_t \lambda_{t+1}^{s} \right] \left( a_{t+1}^s - \tilde{a}_{t+1}^s \right) \right\} - E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left( \lambda_{t}^{m} - \beta E_t \lambda_{t+1}^{m} \right) \left( a_{t+1}^m - \tilde{a}_{t+1}^m \right) \right\}
\]

\[
+ \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi_T^{s} \tilde{a}_{T+1}^{s} + \phi_T^{m} a_{T+1}^{m} \right] \right\} - \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi_T^{s} \tilde{a}_{T+1}^{s} + \phi_T^{m} \tilde{a}_{T+1}^{m} \right] \right\}.
\]

Since \( U \) and \( S \) are concave and differentiable, \( U (c_t) - U (\tilde{c}_t) \geq U' (c_t) (c_t - \tilde{c}_t) \), and

\[
S (\lambda_{t+1} a_{t+1}) + \frac{\partial S (\lambda_{t+1} a_{t+1})}{\partial a_{t+1}^{s}} (\tilde{a}_{t+1}^{s} - a_{t+1}^{s}) + \frac{\partial S (\lambda_{t+1} a_{t+1})}{\partial a_{t+1}^{m}} (\tilde{a}_{t+1}^{m} - a_{t+1}^{m}) \geq S (\lambda_{t+1} \tilde{a}_{t+1}^{s}),
\]

\[

\]

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with \( \frac{\partial S(\lambda_{t+1}a_{t+1})}{\partial a_{t+1}^i} = S'(\lambda_{t+1}a_{t+1})\lambda_{t+1}^j \) for \( i = s, m, \) so

\[
\Delta \geq E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ U'(c_t) - \frac{1}{w_t} \right] (c_t - \tilde{c}_t) \right\} \\
+ E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \beta E_t \left[ 1 + S'(\lambda_{t+1}a_{t+1}) \right] \lambda_{t+1}^s - \frac{1}{w_t} \phi^s_t \right\} (a_{t+1}^s - \tilde{a}_{t+1}^s) \right\} \\
+ E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \beta E_t \left[ 1 + S'(\lambda_{t+1}a_{t+1}) \right] \lambda_{t+1}^m - \lambda^m_t \right\} (a_{t+1}^m - \tilde{a}_{t+1}^m) \right\} \\
+ \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\} - \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\}.
\]

With (8)–(10) and the fact that \( \tilde{c}_t \geq 0, \tilde{a}_{t+1}^s \geq 0, \) and \( \tilde{a}_{t+1}^m \geq 0 \) (because \( \tilde{\chi} \) is feasible), the previous inequality implies

\[
\Delta \geq E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[ U'(c_t) - \frac{1}{w_t} \right] c_t \right\} \\
+ E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \beta E_t \left[ 1 + S'(\lambda_{t+1}a_{t+1}) \right] \lambda_{t+1}^s - \frac{1}{w_t} \phi^s_t \right\} a_{t+1}^s \right\} \\
+ E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left\{ \beta E_t \left[ 1 + S'(\lambda_{t+1}a_{t+1}) \right] \lambda_{t+1}^m - \lambda^m_t \right\} a_{t+1}^m \right\} \\
+ \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\} - \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\}.
\]

Use (8)–(10) once again, to obtain

\[
\Delta \geq \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\} - \liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\}
\]

\[
\geq -\liminf_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi^s_T a^s_{T+1} + \phi^m_T a^m_{T+1} \right] \right\}.
\]

With (11) and (12), this last inequality implies \( \Delta \geq 0, \) so the plan \( \chi \) is optimal.

Step (ii): Next, show that an optimal plan, \( \chi = \{ c_t, x_t, a^s_{t+1}, a^m_{t+1} \}_{t=0}^{\infty} \), must satisfy (8)–(12). Since \( \chi \) is an optimal plan, (23) implies that for each \( t, \) \( c_t = \arg \max_{c \geq 0} \left[ U(c) - \frac{1}{w_t} c \right], \)
and

\[
a_{t+1} \in \arg \max_{a_{t+1} \geq 0} \left\{ -\frac{1}{w_t} \phi^s_t \tilde{a}_{t+1}^s - \lambda^m_t \tilde{a}_{t+1}^m + \beta E_t \left[ S(\lambda_{t+1} \tilde{a}_{t+1}) + \lambda^s_{t+1} \tilde{a}_{t+1}^s + \lambda^m_{t+1} \tilde{a}_{t+1}^m \right] \right\}.
\]
Since both $U$ and $S$ are differentiable, $\{c_t, a_{t+1}\}_{t=0}^\infty$ must satisfy (8)–(10). To show that (11) and (12) are necessary for an optimum, use the optimal plan $\chi = \{c_t, x_t, a_{t+1}\}_{t=0}^\infty$ to construct the feasible plan $\chi^\varepsilon = \{c_t, x_t^\varepsilon, (1 - \varepsilon) a_{t+1}\}_{t=0}^\infty$, for some small $\varepsilon > 0$, where
\[
x_0^\varepsilon = \frac{1}{w_0} \left[ (\phi_0^s + d_0) a_0^m + \phi_0^m (a_0^m + T_0) - (1 - \varepsilon) (\phi_0^s a_1^m + \phi_0^m a_1^m) - a_0 \right],
\]
and
\[
x_t^\varepsilon = \frac{1}{w_t} \left\{ (1 - \varepsilon) \left[ (\phi_t^s + d_t) a_t^s + \phi_t^m a_t^m - \phi_t^s a_{t+1}^s - \phi_t^m a_{t+1}^m \right] - c_t + \phi_t^m T_t \right\}
\]
for $t \geq 1$. Let $\Delta_\varepsilon \equiv U(\chi, a_0, \omega_0) - U(\chi^\varepsilon, a_0, \omega_0)$; then,
\[
\Delta_\varepsilon = E_0 \left\{ \sum_{t=0}^\infty \beta^t \left\{ \beta E_t \left[ S(\lambda_{t+1} a_{t+1}) - S((1 - \varepsilon) \lambda_{t+1} a_{t+1}) \right] - \varepsilon \left[ \frac{1}{w_t} \phi_t^s a_{t+1}^s + \lambda_t^m a_{t+1}^m - \beta E_t \lambda_{t+1} a_{t+1} \right] \right\} \right\}
\]
for $T \to \infty$.

Divide the previous expression by $\varepsilon$, and take the limit as $\varepsilon \to 0$ to arrive at
\[
\lim_{\varepsilon \to 0} \frac{\Delta_\varepsilon}{\varepsilon} = E_0 \left\{ \sum_{t=0}^\infty \beta^t \left\{ - \frac{1}{w_t} \phi_t^s a_{t+1}^s - \lambda_t^m a_{t+1}^m + \beta E_t \left[ 1 + S'(\lambda_{t+1} a_{t+1}) \lambda_{t+1} a_{t+1} \right] \right\} \right\}
\]
and the optimality of $\chi$ requires
\[
0 \leq -\lim_{T \to \infty} E_0 \left\{ \beta^T \frac{1}{w_T} \left[ \phi_T^s a_T^s + \phi_T^m a_T^m \right] \right\}.
\]
Since $\beta^T \frac{1}{w_T} \phi_T^s a_{T+1}^s \geq 0$ and $\beta^T \frac{1}{w_T} \phi_T^m a_{T+1}^m \geq 0$ for all $t$, it follows that (11) and (12) must hold.

**Proof of Proposition 2.** Note that $i_t : \Omega \to \mathbb{R}^+$ is an $\mathcal{F}_t$–measurable function, and from (17), $i_t(\omega) = 0$ a.s. for all $t$ implies $L(\Lambda_{t+1}) = 1$ a.s. for all $t$. Then, since
\[
L(\Lambda_{t+1}) = \begin{cases} 1 & \text{if } q^* \leq \Lambda_{t+1} \\ 1 - \alpha + \alpha u'(\Lambda_{t+1}) & \text{if } \Lambda_{t+1} < q^* \end{cases},
\]
$L(\Lambda_{t+1}) = 1$ a.s. for all $t$, implies $q_{t+1} = \min(\Lambda_{t+1}, q^*) = q^*$ a.s. for all $t$. To conclude, note that $q_{t+1} = q^*$ a.s. for all $t$ implies $q^* \leq \Lambda_{t+1}$ a.s. for all $t$, so (26) implies $L(\Lambda_{t+1}) = 1$ a.s. for all $t$, and (17) implies $i_t(\omega) = 0$ a.s. for all $t$. ■
Proof of Proposition 3. Let $\Omega_i^s = \{ \omega \in \Omega : q^s - \lambda_i^s (\omega) > 0 \}$, and $T = \{ t \in \{0, 1, \ldots \} : E_0 [ \mathbb{I}_{\Omega_t^s} (\omega)] > 0 \}$, where $\mathbb{I}_{\Omega_t^s} (\omega)$ is an indicator function that equals 1 if $\omega \in \Omega_t^s$. Note that $E_0 [ \mathbb{I}_{\Omega_t^s} (\omega)] = \mathbb{P} (\Omega_t^s) \equiv \pi_t$ in the statement of the proposition.

Step 1 ($\Leftarrow$): Show that if (19) and (20) hold, then there exists a monetary equilibrium with $i_t = 0$ a.s. for all $t$. Construct the equilibrium as follows. Set $q_t (\omega) = q^*$ for every $\omega$ and all $t$. Then the Euler equations (13) and (14) become

$$U' (d_t) \phi_t^m = \beta E_t [U' (d_{t+1}) (\phi_{t+1} + d_{t+1})]$$

$$\lambda_t^m = \beta E_t \lambda_{t+1}^m.$$  (27)

Let $\phi_t^s = \phi_t^s^*$, for all $\omega$ and $t$, and notice that it satisfies (15) and (27). Let $\lambda_0^m$ be a positive constant (it will be determined below), let $\lambda_t^m (\omega) = \beta^{-t} \lambda_0^m$ for all $\omega$ and $t$, and notice that $\{\lambda_t^m\}_{t=0}^\infty$ satisfies (28). Also,

$$\liminf_{t \to \infty} E_0 [\beta^t \lambda_t^m M_{t+1}] = \liminf_{t \to \infty} E_0 [\lambda_0^m M_{t+1}] = \lambda_0^m \liminf_{t \to \infty} M_{t+1} = 0,$$

where the last equality follows from condition (19), so (16) is also satisfied. All that remains is to show that $\lambda_0^m$ can be chosen such that $\lambda_t^m (\omega) > 0$ for all $\omega$ and $t$ (so the equilibrium constructed is indeed monetary), and such that $\lambda_t^m (\omega) M_t \geq q^* - \lambda_t^s (\omega)$ a.s. for all $t$ (so that real balances are consistent with $q_t = q^*$, and hence with $i_t = 0$ a.s. for all $t$). Given that $\lambda_t^m (\omega) = \beta^{-t} \lambda_0^m$, any positive choice of $\lambda_0^m$ guarantees $\lambda_t^m (\omega) > 0$ for every $\omega$ and all $t$. In particular, if the primitives of the economy are such that $T = \emptyset$, choose $\lambda_0^m = k \in (0, \infty)$, which implies $\lambda_t^m (\omega) M_t = k M_t \beta^{-t} > 0 \geq q^* - \lambda_t^s (\omega)$ for all $\omega$ and $t$. Conversely, if $T \neq \emptyset$, choose

$$\lambda_0^m = \frac{q^*}{\inf_{t \in T} M_t \beta^{-t}} \in (0, \infty).$$  (29)

Then

$$\lambda_t^m (\omega) M_t = \frac{M_t \beta^{-t}}{\inf_{t \in T} M_t \beta^{-t}} q^* \geq q^* - \lambda_t^s (\omega) \quad \text{for all } \omega, t.$$

Step 2 ($\Rightarrow$): Show that if $\{q_t, \phi_t^s, \phi_t^m\}_{t=0}^\infty$ is a monetary equilibrium with $i_t = 0$ a.s. for all $t$, then (19) and (20) must hold. In such an equilibrium, $\{\lambda_t^m\}$ satisfies (28), which together with the Law of Iterated Expectations, implies

$$\lambda_0^m = E_0 [\beta^t \lambda_t^m].$$  (30)

Also, in any monetary equilibrium,

$$\liminf_{t \to \infty} E_0 [\beta^t \lambda_t^m M_{t+1}] = \lambda_0^m \liminf_{t \to \infty} M_{t+1} = 0$$  (31)
by (16). Since \( \lambda_0^m > 0 \) in a monetary equilibrium, (31) implies (19). A monetary equilibrium with \( i_t = 0 \) a.s. for all \( t \) also satisfies

\[
\lambda_t^m(\omega) M_t \begin{cases} 
\geq q^* - \lambda_t^{s*}(\omega) & \text{for all } t \text{ and a.e. } \omega \in \Omega_t^* \\
> 0 & \text{for all } t \text{ and a.e. } \omega \not\in \Omega_t^*. 
\end{cases} 
\]  

(32)

Multiply the top inequality in (32) through by \( I_{\Omega_t^*}(\omega) \), and the bottom inequality by \( [1 - I_{\Omega_t^*}(\omega)] \), and add them to obtain

\[
\lambda_t^m(\omega) M_t \geq I_{\Omega_t^*}(\omega) [q^* - \lambda_t^{s*}(\omega)] \quad \text{for all } t \text{ and a.e. } \omega. 
\]  

(33)

For all \( t \notin T \), (33) implies \( \lambda_t^m(\omega) \geq 0 \) for a.e. \( \omega \), merely an equilibrium condition. But (33) also implies

\[
\lambda_t^m(\omega) M_t \geq I_{\Omega_t^*}(\omega) (q^* - \bar{\lambda}^{**}) \quad \text{for all } t \in T \text{ and a.e. } \omega, 
\]

where \( \bar{\lambda}^{**} \equiv \sup_{t \in T} \sup_{\omega \in \Omega_t^*} \lambda_t^{s*}(\omega) \). Together with (30), this last inequality implies

\[
\lambda_0^m M_t \beta^{-t} \geq \pi_t (q^* - \bar{\lambda}^{**}) \quad \text{for all } t \in T. 
\]

This last condition is vacuous if the primitives of the economy are such that \( T = \emptyset \), but implies

\[
\inf_{t \in T} M_t \beta^{-t} \geq \frac{q^* - \bar{\lambda}^{**}}{\lambda_0^m} \inf_{t \in T} \pi_t > 0 \quad \text{if } T \neq \emptyset. 
\]

Hence, (20) is also necessary in a monetary equilibrium with \( i_t = 0 \) a.s. for all \( t \).
References


